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Complex Foliated Tori and their Moduli Spaces

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Abstract. We consider complex foliated tori, their periods and polarizations on them. We define the moduli space of polarized complex foliated tori and show that it is a normal analytic complex space. Finally we discuss some examples.

Introduction

Complex foliated tori are manifolds of the form $\mathbb{T} = \mathbb{C}^n \times \mathbb{R}^k/\Lambda$, where $\Lambda \subset \mathbb{R}^{2n+k}$ is a discrete subgroup of maximal rank (i.e. a lattice) equipped with the complex foliation induced by the projection $\mathbb{C}^n \times \mathbb{R}^k \to \mathbb{R}^k$. They appeared, as far as we know, in [1] in the context of *quasi-abelian varieties*, namely the toroidal groups $\mathbb{C}^n/\Lambda$ where Λ is a discrete subgroup, not necessarily of maximal rank (see also [2], [5]). Indeed, every quasi-abelian variety factorizes as the product of a complex foliated torus by a real vector space ([1], §4). They were also considered, in the theory of CR-bundles over complex foliated manifolds, in [8].

The aim of the present paper is to introduce and study the moduli space $\mathcal{M}_{n,k}^{\omega}$ of *polarized complex foliated tori,* i.e. of complex foliated tori endowed with a fixed polarization ω (see Sections 2, 3). Our main result is the following:

$\mathcal{M}_{n,k}^{\omega}$ *is a normal complex space of real dimension* $n(n+2k+1)$.

In order to prove it, after fixing some notations, we introduce the notion of polarization on a complex foliated torus, and we characterize it in terms of period matrices, thus stating the analogous of Riemann conditions (see §2). Next, we describe the space of periods of polarized complex foliated tori in terms of complex homogeneous spaces (see §3), namely, we introduce the space $\mathcal{Z}_\omega$ of matrices of the form $\begin{pmatrix} J & 0 \\ L & I \end{pmatrix}$, where $J \in Sp^\omega_{2n}(\mathbb{R})$ is a complex structure and $L \in M_{k,2n}(\mathbb{R})$ an arbitrary matrix. The group

$$G_\omega = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & I \end{pmatrix} \middle| \alpha \in Sp^\omega_{2n}(\mathbb{R}),\, \gamma \in M_{k,2n}(\mathbb{R}) \right\}$$

acts transitively on $\mathcal{Z}_\omega$ and, like in the case of abelian varieties (cf. [3], [6]), it turns out that any discrete subgroup of G acts on $\mathcal{Z}$ in a properly discontinuous way (Theorem 4.4).

As a consequence of a classical theorem of Henri Cartan [4], we obtain that the moduli space of polarized foliated tori inherits a structure of normal complex space.

1 Preliminaries

Consider the real vector space $\mathbb{C}^n \times \mathbb{R}^k$ with its canonical CR-structure. A smooth CR-map $\mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^n \times \mathbb{R}^k$ is a smooth map of the form

$$\begin{cases} z' = f(z,t) \\ t' = g(t) \end{cases}$$

where f is holomorphic w.r.t. z. Points of $\mathbb{C}^n \times \mathbb{R}^k$ are written in the form (z,t) with $z \in \mathbb{C}^n$ and $t \in \mathbb{R}^k$.

Let Λ be a lattice of rank $2n+k$ in the vector space $\mathbb{R}^{2n+k} \cong \mathbb{C}^n \times \mathbb{R}^k$. The covering map $\mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^n \times \mathbb{R}^k/\Lambda$ defines a natural CR-structure on the quotient space.

$\mathbb{T} := \mathbb{C}^n \times \mathbb{R}^k/\Lambda$ equipped with this CR-structure is called a *complex foliated torus.*

A complex foliated torus is in particular a CR-Lie group, i.e. a Lie group with a CR-structure such that the group operations are compatible with this structure. A *morphism* between complex foliated tori is a morphism of CR-Lie groups.

Complex foliated tori can be described in terms of *periods.* Consider a $\mathbb{Z}$-basis $\{Z_1 \oplus R_1, ..., Z_{2n+k} \oplus R_{2n+k}\}$ of the lattice Λ (where $Z_i \in \mathbb{C}^n$ and $R_i \in \mathbb{R}^k$). By definition, the $(n+k) \times (2n+k)$ complex matrix

$$\Omega = \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_{2n+k} \\ R_1 & R_2 & \cdots & R_{2n+k} \end{pmatrix}$$

is called a *period matrix* of the foliated torus induced by Λ.

Any linear isomorphism φ of the form

$$\begin{pmatrix} z \\ t \end{pmatrix} \mapsto \begin{pmatrix} Az + Bt \\ Dt \end{pmatrix}$$

(with $A \in GL_n(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$ and $D \in GL_k(\mathbb{R})$) induces an isomorphism $\mathbb{C}^n \times \mathbb{R}^k/\Lambda \cong \mathbb{C}^n \times \mathbb{R}^k/\Lambda'$ of complex foliated tori, where $\Lambda' = \varphi(\Lambda)$.

We denote by $L_{n,k}$ the group of matrices

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A, B, D are as above.

Remark 1.1 *Linear Algebra arguments show that φ can be chosen in such a way that the torus $\mathbb{C}^n \times \mathbb{R}^k/\Lambda'$ has a period matrix of the form*

$$\Omega = \begin{pmatrix} Z_1 & \cdots & Z_{2n} & 0 & \cdots & 0 \\ R_1 & \cdots & R_{2n} & e_1 & \cdots & e_k \end{pmatrix}$$

$\{e_1, ..., e_k\}$ *being the canonical basis of* $\mathbb{R}^k$.

We stress that not every element of $M_{n+k,2n+k}(\mathbb{C})$ is a period matrix for a complex foliated torus. Since a matrix

$$\Omega = \begin{pmatrix} Z_1 & \cdots & Z_{2n} & 0 & \cdots & 0 \\ R_1 & \cdots & R_{2n} & e_1 & \cdots & e_k \end{pmatrix}$$

has rank $n + k$ if and only if

$$\det \begin{pmatrix} Z_1 & \cdots & Z_{2n} & 0 & \cdots & 0 \\ \overline{Z}_1 & \cdots & \overline{Z}_{2n} & 0 & \cdots & 0 \\ R_1 & \cdots & R_{2n} & e_1 & \cdots & e_k \end{pmatrix} \neq 0$$

by the previous remark we get that a matrix

$$\Omega = \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_{2n+k} \\ R_1 & R_2 & \cdots & R_{2n+k} \end{pmatrix}$$

is the period matrix for a complex foliated torus if and only if

$$\det \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_{2n+k} \\ \overline{Z}_1 & \overline{Z}_2 & \cdots & \overline{Z}_{2n+k} \\ R_1 & R_2 & \cdots & R_{2n+k} \end{pmatrix} \neq 0$$

Let $\mathbb{T}$ and $\mathbb{T}'$ be two complex foliated tori. A CR-map $\varphi : \mathbb{T} \to \mathbb{T}'$ is induced by a CR-map $\tilde{\varphi} : \mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^n \times \mathbb{R}^k$ of the form

$$\tilde{\varphi}(z,t) = (Az + Bt + h(t), Dt + k(t))$$

where A, B and D are constant matrices, h and k are smooth Λ-invariant maps (see [8], Proposition 6.6).

As a corollary (see [8], Corollary 6.7), we obtain the following: let $\mathbb{T}$ and $\mathbb{T}'$ be complex foliated tori and Ω and Ω' be period matrices for $\mathbb{T}$ and $\mathbb{T}'$ respectively. Then $\mathbb{T}$ and $\mathbb{T}'$ are isomorphic (as CR-Lie groups) if and only if

$$M\Omega = \Omega'\gamma \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with $A \in GL_n(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$, $D \in GL_k(\mathbb{R})$ and $\gamma \in GL_{2n+k}(\mathbb{Z})$.

Accordingly, two period matrices Ω and Ω' are said to be *equivalent* if the above condition is fulfilled. By definition, the *moduli space* of all complex foliated tori is the quotient $\mathcal{M}_{n,k}$ of the space of period matrices modulo the equivalence relation (1.1).

$\mathcal{M}_{n,k}$ can be also described as a double quotient

$$L_{n,k} \backslash GL_{2n+k}(\mathbb{R}) / GL_{2n+k}(\mathbb{Z}).$$

For this, consider the space $\mathcal{P}$ of all period matrices modulo the action $\Omega \mapsto M\Omega$, $M \in L_{n,k}$. The group GL_{2n+k} acts transitively on $\mathcal{P}$ (the map $\Omega \mapsto \Omega A^{-1}$, $A \in GL_{2n+k}$ descends to the quotient $\mathcal{P}$) and the isotropy subgroup is just $L_{n,k}$. Therefore, $\mathcal{P} \cong L_{n,k} \backslash GL_{2n+k}(\mathbb{R})$ and consequently, $\mathcal{M}_{n,k} \cong \mathcal{P}/GL_{2n+k}(\mathbb{Z})$. This identification shows that, when $n > 1$, $\mathcal{M}_{n,k}$ is not even a Hausdorff space. Indeed, the same arguments of [3] Corollary VII.2.5 can be used to check that the space $\mathcal{P}/\Gamma$, where Γ is any discrete subgroup of $L_{n,k}$, actually is not even a T_1 space. The only point to remark is that our group $L_{n,k}$ is reductive, since it is an abelian extension of the reductive group $GL_{2n}(\mathbb{R})$.

2 An example: $\mathcal{M}_{1,1}$

In this section we will study the simplest case of the moduli space of complex foliated tori, namely we take $n = 1$ and $k = 1$. We will slightly change our point of view, giving an explicit description of the moduli space $\mathcal{M}_{1,1}$.

Let us consider a torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ equipped with a complex linear foliation $\mathcal{F}$, thus induced by a complex foliation of parallel planes in $\mathbb{R}^3$. Let Π

be the plane through the origin of $\mathbb{R}^3$ and $J : \Pi \to \Pi$ be the corresponding complex structure.

Let us forget the complex structure for a while: given $\mathcal{F}$, of course the set $\Gamma_{\mathcal{F}}$ of lines through the origin of Π is a $\mathbb{RP}^1$ inside $\mathbb{RP}^2$. By taking the complexification of the inclusion $\Gamma_{\mathcal{F}} = \mathbb{RP}^1 \subset \mathbb{RP}^2$, we get $\Gamma_{\mathcal{F}}^{\mathbb{C}} = \mathbb{CP}^1 \subset \mathbb{CP}^2$, with $\Gamma_{\mathcal{F}}^{\mathbb{C}} \cap \mathbb{RP}^2 = \Gamma_{\mathcal{F}}$ and $\Gamma_{\mathcal{F}}^{\mathbb{C}} \setminus \Gamma_{\mathcal{F}} =$
$Ch^+ \cup \mathcal{H}^-$ (Poincaré half-planes).

Next, we turn to the complex structure J and to the induced complexification $J^{\mathbb{C}} : \Pi^{\mathbb{C}} \to \Pi^{\mathbb{C}}$. The former acts freely (without fixed point) as an involution on $\Gamma_{\mathcal{F}}$, the latter acts as an involution on $\Gamma_{\mathcal{F}}^{\mathbb{C}}$, with two fixed points, corresponding to the eigenspaces relative to the eigenvalues $\pm i$. The choice of the eigenspace relative to i determines a map $J \mapsto p \in \Gamma_{\mathcal{F}}^{\mathbb{C}} \setminus \Gamma_{\mathcal{F}}$, which is bijective, since J is real (so that the eigenspace relative to $-i$ is determined by the one corresponding to $+i$).

Therefore, the set of complex structures on Π can be identified with $\Gamma_{\mathcal{F}}^{\mathbb{C}} \setminus \Gamma_{\mathcal{F}}$. Now, observe that, for any point $p \in \mathbb{CP}^2 \setminus \mathbb{RP}^2$, there exists a unique complex projective line $\Gamma_p^{\mathbb{C}} \subset \mathbb{CP}^2$, passing through p, which is also the complexification of a real projective line $\Gamma_p \in \mathbb{RP}^2$.

Hence, the period space $\mathcal{P}$ is naturally identified with $\mathbb{CP}^2 \setminus \mathbb{RP}^2$.

Remark 2.1 *The map $p \mapsto \Gamma_p$ defined above is a smooth fibration $\mathbb{CP}^2 \setminus \mathbb{RP}^2 \to (\mathbb{RP}^2)\check{}$, whose fiber is $\mathcal{H}^+ \cup \mathcal{H}^-$, which just forgets the complex structure J.*

Coming back to the moduli space, we consider the standard $GL_3(\mathbb{Z})$-action on $\mathbb{CP}^2 \setminus \mathbb{RP}^2$: two complex foliations $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{T}^3$ are CR-isomorphic if and only if there exists $A \in GL_3(\mathbb{Z})$ such that

1) $A\Pi \subset \Pi$.
2) $A|_{\Pi} \circ J = J' \circ A|_{\Pi}$.

Therefore, the moduli space is

$$\mathcal{M}_{1,1} = \mathbb{CP}^2 \setminus \mathbb{RP}^2 / GL_3(\mathbb{Z})$$

The $GL_3(\mathbb{Z})$-action, accordingly to what we observed at the end of the previous section, is not properly discontinuous, since, for example, given any $A \in GL_3(\mathbb{Z})$, we have $A\mathcal{C} \cap \mathcal{C} \neq \emptyset$, where $\mathcal{C}$ is the standard conic $z_0^2 + z_1^2 + z_2^2 = 0$ in $\mathbb{CP}^2$.

In order to define a reasonable moduli space, we have to introduce an additional datum on complex foliated tori, i.e. a *polarization*, and, accordingly, to study the equivalence of complex foliated tori endowed with this enriched structure.

3 Polarizations and Riemann Conditions

Let $\mathbb{T} = \mathbb{C}^n \times \mathbb{R}^k/\Lambda$ be a complex foliated torus. A cohomology class $[\omega] \in H^2(\mathbb{T}, \mathbb{Z})$ is represented by a differential form ω on $\mathbb{R}^{2n} \times \mathbb{R}^k$ with constant coefficients, which is integral on Λ.

Definition 3.1 *We say that* $[\omega]$ *is a* polarization *if*

(i) ω *is of rank* $2n$*;*

(ii) $\omega_{|_{\mathbb{R}^{2n}\times\{0\}}}$ *is the imaginary part of a definite positive Hermitian form on* $\mathbb{C}^n$*.*

A *polarized complex foliated torus* is a pair $(\mathbb{T}, [\omega])$ where $\mathbb{T}$ is a complex foliated torus and $[\omega]$ a polarization on it. Two polarized foliated tori $(\mathbb{T}, [\omega])$, $(\mathbb{T}', [\omega'])$ are said to be *isomorphic*, if there exists an isomorphism $\varphi : \mathbb{T} \to \mathbb{T}'$ such that $[\varphi^*\omega'] = [\omega]$.

In order to characterize the classes $\omega \in H^2(\mathbb{T}, \mathbb{Z})$ which determine a polarization, we consider an adapted period matrix

$$\Omega = \begin{pmatrix} Z_1 & \dots Z_{2n} & 0 & \dots & 0 \\ R_1 & \dots R_{2n} & e_1 & \dots & e_k \end{pmatrix} = \begin{pmatrix} C & 0 \\ R & I \end{pmatrix}$$

where $C \in M_{n,2n}(\mathbb{C})$, $R \in M_{k,2n}(\mathbb{R})$ and $\{e_1, \dots, e_k\}$ is the standard basis of $\mathbb{R}^k$. Let $(x_1, \dots, x_{2n}, y_1 \dots, y_k)$ be the coordinates on $\mathbb{R}^{2n} \times \mathbb{R}^k$ associated to the lattice basis $\{Z_1 \oplus R_1, ..., Z_{2n+k} \oplus R_{2n+k}\}$. If $(z_1, \dots, z_n, \xi_1, \dots, \xi_k)$ denote the standard coordinates on $\mathbb{C}^n \times \mathbb{R}^k$, then we have

$$\begin{cases} dz_\alpha = \sum_A c_{\alpha A} dx_A \\ d\overline{z}_\alpha = \sum_A \overline{c}_{\alpha A} dx_A \\ d\xi_i = \sum_A r_{jA} dx_A + dy_i \end{cases}$$

(where $C = (c_{\alpha B})$, $R = (r_{iA})$ are the blocks of Ω) and capital (small) latin indexes run from 1 to $2n$ (1 to k), and greek indexes from 1 to n.

Let Π be the inverse of $\begin{pmatrix} C & 0 \\ \overline{C} & 0 \\ R & I \end{pmatrix}$, which is therefore of the form

$$\Pi = \begin{pmatrix} \pi & \overline{\pi} & 0 \\ S & \overline{S} & I \end{pmatrix}$$

with $\pi \in M_{2n,n}(\mathbb{C})$ and $S \in M_{k,n}(\mathbb{C})$. We have

$$\begin{cases} dx_A = \sum_\alpha \pi_{A\alpha} dz_\alpha + \sum_\alpha \overline{\pi}_{A\alpha} d\overline{z}_\alpha \\ dy_i \;= \sum_\alpha s_{i\alpha} dz_\alpha + \sum_\alpha \overline{s}_{i\alpha} d\overline{z}_\alpha + d\xi_i \end{cases}$$

(where $\pi = (\pi_{A\alpha})$ and $S = (s_{i\alpha})$ are the blocks of Π.)

Now suppose to have a polarization ω on a complex foliated torus,

$$\omega = \sum_{A<B} \omega_{AB} dx_A \wedge dx_B + \sum_{A,i} \omega_{Ai} dx_A \otimes dy_i + \\ - \sum_{A,i} \omega_{Ai} dy_i \otimes dx_A + \sum_{i<j} \omega_{ij} dy_i \wedge dy_j$$

where $\omega_{AB}, \omega_{Ai}, \omega_{ij} \in \mathbb{Z}$.

Let us define the matrices $\omega^{(1)} = (\omega_{AB})$, $\omega^{(2)} = (\omega_{Ai})$ and $\omega^{(3)} = (\omega_{ij})$ so that the form ω is represented by $\begin{pmatrix} \omega^{(1)} & \omega^{(2)} \\ -{}^t\omega^{(2)} & \omega^{(3)} \end{pmatrix}$. Then, by a direct computation, one shows that condition (ii) of Definition 3.1 is equivalent to

$$\text{(R)} \qquad \begin{cases} {}^t\pi\, \omega^{(1)} \pi + {}^tS\, \omega^{(3)} S + {}^t\pi\, \omega^{(2)} S - {}^tS\, {}^t\omega^{(2)} \pi = 0 \\ \dfrac{1}{\sqrt{-1}} \left({}^t\pi\, \omega^{(1)} \overline{\pi} + {}^tS\, \omega^{(3)} \overline{S} + {}^t\pi\, \omega^{(2)} \overline{S} - {}^tS\, {}^t\omega^{(2)} \overline{\pi} \right) > 0 \end{cases}$$

These relations constitute the foliated version of the classical Riemann conditions.

More coincisely,

Theorem 3.2 *Let $\mathbb{T}$ be a complex foliated torus. Then a cohomology class $[\omega] \in H^2(\mathbb{T}, \mathbb{Z})$ is a polarization on $\mathbb{T}$ if and only if there exists a positive definite Hermitian matrix H such that*

$$\frac{1}{\sqrt{-1}} {}^t\,\Pi \omega \overline{\Pi} = \begin{pmatrix} H & 0 & * \\ 0 & -{}^tH & * \\ * & * & * \end{pmatrix}$$

and

$$\begin{pmatrix} H & 0 & * \\ 0 & -{}^tH & * \\ * & * & * \end{pmatrix}$$

is of rank $2n$.

Remark 3.3 *The above definition of polarization on a complex foliated torus is stronger than the following one: there exists a bilinear skew-symmetric form E on $\mathbb{R}^{2n} \times \mathbb{R}^k$ such that $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and the restriction of E to $\mathbb{R}^{2n} \times \{0\}$ is the imaginary part of a positive definite Hermitian form H on $\mathbb{C}^n \times \{0\}$. Under this condition, in [8], Theorem 6.9, it was proven that $\mathbb{T}$ can be embedded as a CR-manifold in a complex projective space.*

Therefore

Corollary 3.4 *A complex foliated torus $\mathbb{T}$ can be embedded, as a CR-submanifold, in a complex projective space if there exists a cohomology class $[\omega] \in H^2(\mathbb{T}, \mathbb{Z})$ such that Riemann conditions (R) are satisfied.*

Remark 3.5 *We observe that in the case $n = 1$, namely when leaves are complex curves, foliated Riemann conditions (R) are always satisfied. Indeed the period matrix of $\mathbb{T} = \mathbb{C} \times \mathbb{R}^k/\Lambda$ is of the form*

$$\Omega = \begin{pmatrix} z & w & 0 \\ R_1 & R_2 & I \end{pmatrix}$$

where $z, w \in \mathbb{C}$ and $R_1, R_2 \in M_{k,1}(\mathbb{R})$. Now the inverse of

$$\begin{pmatrix} z & w & 0 \\ \overline{z} & \overline{w} & 0 \\ R_1 & R_2 & I \end{pmatrix}$$

is the matrix

$$\Pi = \begin{pmatrix} \pi & \overline{\pi} & 0 \\ S & \overline{S} & I \end{pmatrix}$$

where $\pi \in \mathbb{C}^2$ and $S \in M_{k,1}(\mathbb{C})$. Now let

$$\omega = \begin{pmatrix} \omega^{(1)} & \omega^{(2)} \\ -{}^t\omega^{(2)} & \omega^{(3)} \end{pmatrix}$$

be a constant 2-form taking integer values on the lattice Λ. The first Riemann condition

$${}^t\pi\, \omega^{(1)} \pi + {}^tS\, \omega^{(3)} S + {}^t\pi\, \omega^{(2)} S - {}^tS\, {}^t\omega^{(2)} \pi = 0$$

is trivially satisfied, since $\omega^{(1)}$ and $\omega^{(3)}$ are skew-symmetric, and $\pi \in \mathbb{C}^2$ and $S \in \mathbb{C}^k$ are column vectors; furthermore, ${}^t\pi\, \omega^{(2)} S = {}^tS\, {}^t\omega^{(2)} \pi$ is a complex number.

As far as the second Riemann condition is concerned, it suffices to take $\omega^{(2)} = 0$ and $\omega^{(3)} = 0$ for instance, to fulfil it.

Remark 3.6 *Let $(\mathbb{T}, [\omega])$ be a polarized foliated torus. The kernel of any representative form ω of the polarization $[\omega]$ is completely integrable and transversal to the foliation on $\mathbb{T}$. It is immediate to check that, up to isomorphisms, we may assume that*

$$\Omega = \begin{pmatrix} Z & W & 0 \\ R & S & I \end{pmatrix}$$

where $Z, W \in M_{n,n}(\mathbb{C})$ and $R, S \in M_{k,n}(\mathbb{R})$ and the form ω is represented, with respect to the $\mathbb{Z}$-basis of Λ, given by the columns of Ω, by the matrix

$$\omega = \begin{pmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\Delta = \mathrm{diag}(d_1, \ldots, d_n)$, $d_1, ..., d_n \in \mathbb{Z}^+$ (cf. e.g. [7]).

4 Moduli of polarized foliated tori

Let $\mathbb{T}$ be a polarized torus with a period matrix Ω and a fixed polarization ω as in the above remark. Ω defines a $\mathbb{R}$-isomorphism $\mathbb{R}^{2n+k} \to \mathbb{C}^n \times \mathbb{R}^k$ (also denoted by Ω) fixing $\{0\} \times \mathbb{R}^k$ and sending the standard lattice $\mathbb{Z}^{2n+k}$ into Λ. Let $P : \mathbb{R}^{2n+k} \to \mathbb{R}^{2n+k}$ be defined by

$$\begin{tikzcd} \mathbb{R}^{2n+k} \arrow[r, "\Omega"] \arrow[d, "P"] & \mathbb{C}^n \times \mathbb{R}^k \arrow[d, "i \times \mathrm{id}_{\mathbb{R}^k}"] \\ \mathbb{R}^{2n+k} \arrow[r, "\Omega"] & \mathbb{C}^n \times \mathbb{R}^k \end{tikzcd}$$

where $i \times \mathrm{id}_{\mathbb{R}^k}(z,t) = (iz, t)$. The matrix of P is of the form

$$P = \begin{pmatrix} J & 0 \\ L & I_k \end{pmatrix}$$

with $J^2 = -\mathrm{id}_{\mathbb{R}^{2n}}$ and (by a direct computation) $J \in Sp^{\omega}_{2n}(\mathbb{R})$, and $L \in M_{k,2n}(\mathbb{R})$.

The isomorphisms P determine the structure of polarized foliated torus. We will denote by $\mathcal{Z}_\omega$ the space of such matrices P, by G_ω the group of matrices of the form

$$\begin{pmatrix} X & 0 \\ Y & I_k \end{pmatrix}$$

where $X \in Sp^{\omega}_{2n}(\mathbb{R})$ and $Y \in M_{k,2n}(\mathbb{R})$, and by H_ω its subgroup (isomorphic to U_n) of matrices of the form

$$\begin{pmatrix} U & 0 \\ 0 & I_k \end{pmatrix}$$

with $U \in U_n$.

Let $\mathbb{T}$ and $\mathbb{T}'$ be two complex foliated tori equipped with the same polarization $[\omega]$. We may assume that Ω and Ω' are adapted period matrices for $\mathbb{T}$ and $\mathbb{T}'$ respectively, and that the polarization $[\omega]$ is represented by a same constant form ω as in Remark 3.6. Then, by definition, $\mathbb{T}$ is isomorphic to $\mathbb{T}'$ if and only if there exists a CR-isomorphism $\varphi : \mathbb{T} \to \mathbb{T}'$ preserving $\ker \omega$ and which is symplectic on leaves, namely

$$\Omega \sim \Omega' \iff \exists\, M \in H_\omega \quad \exists\, \gamma \in G_\omega \cap SL_{2n+k}(\mathbb{Z}) \qquad M\Omega = \Omega'\gamma$$

(cf. (1.1)). On the space $\mathcal{Z}_\omega$ the equivalence is given by

$$P \sim P' \iff \exists\, \gamma \in G_\omega \cap SL_{2n+k}(\mathbb{Z}) \qquad P' = \gamma^{-1} P \gamma$$

The *moduli space* of polarized foliated tori is then defined by

$$\mathcal{M}^{\omega}_{n,k} := \mathcal{Z}_\omega / \sim$$

En passant, we observe that there is an obvious fibration $\mathcal{Z}_\omega \to \mathcal{H}_\omega$ over the Poincaré–Siegel half-plane, given by

$$\begin{pmatrix} J & 0 \\ L & I \end{pmatrix} \longmapsto J$$

Theorem 4.1 *$\mathcal{Z}_\omega$ is a homogeneous space isomorphic to G_ω / H_ω*

PROOF: The group G_ω acts in a natural way on $\mathcal{Z}_\omega$ by conjugation:

$$(M, P) \mapsto MPM^{-1}$$

One can check that such an action is transitive, and that the isotropy subgroup at the point

$$P_0 = \begin{pmatrix} J_0 & 0 \\ 0 & I \end{pmatrix}$$

(J_0 being the standard complex structure on $\mathbb{R}^{2n}$) is given by H_ω.

QED

Corollary 4.2 $\dim_{\mathbb{R}} \mathcal{Z}_\omega = n(n+2k+1)$.

Observe that $\mathcal{Z}_\omega = G_\omega/H_\omega$ is a *reductive homogeneous space,* consequently the Lie algebra $\mathfrak{g}_\omega$ =Lie(G_ω) admits a decomposition of the form $\mathfrak{g}_\omega = \mathfrak{h}_\omega + \mathfrak{m}_\omega$, where $\mathfrak{h}_\omega$ =Lie(H_ω) and $\mathfrak{m}_\omega$ is a vector space in $\mathfrak{g}_\omega$ such that $\mathfrak{h}_\omega \cap \mathfrak{m}_\omega = \{0\}$ and $\mathrm{ad}(H_\omega)\mathfrak{m}_\omega \subset \mathfrak{m}_\omega$.

Indeed it suffices to consider

$$\mathfrak{m}_\omega = \left\{ \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ \alpha & \beta & 0 \end{pmatrix} \middle| A, B \in M_{n,n}(\mathbb{R}),\ A = {}^tA\,,\ B = {}^tB\,,\ \alpha, \beta \in M_{k,n}(\mathbb{R}) \right\}$$

Now, since

$$\mathfrak{g}_\omega = \left\{ \begin{pmatrix} K & 0 \\ Q & 0 \end{pmatrix} \middle| K \in \mathfrak{sp}^\omega_{2n}(\mathbb{R}),\ Q \in M_{k,2n}(\mathbb{R}) \right\}$$

and

$$\mathfrak{h}_\omega = \left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \middle| U \in \mathfrak{u}_n \right\}$$

we have that $\mathfrak{g}_\omega = \mathfrak{h}_\omega \oplus \mathfrak{m}_\omega$ as a vector space, and $[\mathfrak{h}_\omega, \mathfrak{m}_\omega] \subset \mathfrak{m}_\omega$, so that (being H_ω connected) $\mathrm{ad}(H_\omega)\mathfrak{m}_\omega \subset \mathfrak{m}_\omega$. Of course we consider $\mathfrak{gl}_n(\mathbb{C})$ as embedded in $\mathfrak{gl}_{2n}(\mathbb{R})$.

Moreover we can prove that $\mathcal{Z}_\omega$ is actually a Kähler homogenous space:

Theorem 4.3 *$\mathcal{Z}_\omega$ is a Kähler manifold in a natural way.*

PROOF: Since $\mathcal{Z}_\omega$ is a reductive homogeneous space, in view [9] §10, to define an invariant integrable complex structure on $\mathcal{Z}_\omega$, it suffices to define an integrable Koszul operator, thus a linear endomorphism $\mathbb{J} : \mathfrak{m}_\omega \to \mathfrak{m}_\omega$ such that

(1) $\mathbb{J}^2 = -I$;

(2) $\forall Y \in \mathfrak{h}_\omega \qquad \mathbb{J} \circ \mathrm{ad}_Y = \mathrm{ad}_Y \circ \mathbb{J}$

(3) $\forall \xi, \eta \in \mathfrak{m}_\omega \qquad [\mathbb{J}\xi, \mathbb{J}\eta]_{\mathfrak{m}_\omega} - [\xi, \eta]_{\mathfrak{m}_\omega} - \mathbb{J}[\xi, \mathbb{J}\eta]_{\mathfrak{m}_\omega} - \mathbb{J}[\mathbb{J}\xi, \eta]_{\mathfrak{m}_\omega} = 0$

(where $[X, Y]_{\mathfrak{m}_\omega}$ is the projection of $[X, Y]$ onto the factor $\mathfrak{m}_\omega$).

In our case, for any

$$\xi = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ \alpha & \beta & 0 \end{pmatrix} \in \mathfrak{m}_\omega$$

we put

$$\mathbb{J}\begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ \alpha & \beta & 0 \end{pmatrix} := \begin{pmatrix} -B & A & 0 \\ A & B & 0 \\ -\beta & \alpha & 0 \end{pmatrix}$$

Clearly $\mathbb{J}^2 = -I$. To check condition (2), we set

$$Y = \begin{pmatrix} U & V & 0 \\ -V & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}_\omega$$

By definition:

$$\mathbb{J}[Y, \xi] = \begin{pmatrix} AV + VA - [U, B] & [U, A] + VB + BV & 0 \\ [U, A] + VB + BV & [U, B] - AV - VA & 0 \\ \alpha V + \beta U & \beta V - \alpha U & 0 \end{pmatrix} = [Y, \mathbb{J}\xi]$$

As for the integrability condition (3), a direct computation shows that, for any $\xi, \eta \in \mathfrak{m}$

$$[\mathbb{J}\xi, \mathbb{J}\eta] - [\xi, \eta] - \mathbb{J}[\xi, \mathbb{J}\eta] - \mathbb{J}[\mathbb{J}\xi, \eta] = 0$$

Now we introduce the metric: again, in view of [9] §10, recall that there is a bijective correspondence between G_ω-invariant metrics on $\mathcal{Z}_\omega$ and ad_{H_ω}-invariant scalar products on $\mathfrak{m}_\omega$. Let us fix the following scalar product on $\mathfrak{m}_\omega$:

$$\langle \xi, \eta \rangle = \mathrm{tr}\, {}^t\xi\, \eta$$

(for each $\xi, \eta \in \mathfrak{m}_\omega$.) Since $\langle , \rangle$ is ad_{H_ω}-invariant, it induces a G_ω-invariant metric g on $\mathcal{Z}_\omega$.

Now, a direct computation shows that $\mathbb{J}$ is an isometry with respect to the invariant metric g, i.e. $\langle \mathbb{J}\xi, \mathbb{J}\eta \rangle = \langle \xi, \eta \rangle$.

Finally, to show that g is a Kähler metric, we compute the exterior derivative of the Kähler form $\kappa(\xi, \eta) = \langle \mathbb{J}\xi, \eta \rangle$. Given

$$\xi_1 = \begin{pmatrix} A_1 & B_1 & 0 \\ B_1 & -A_1 & 0 \\ \alpha_1 & \beta_1 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} A_2 & B_2 & 0 \\ B_2 & -A_2 & 0 \\ \alpha_2 & \beta_2 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} A_3 & B_3 & 0 \\ B_3 & -A_3 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}$$

we have

$$(\star) \qquad d\kappa(\xi_1, \xi_2, \xi_3) = \frac{1}{3!}\Big(- \kappa([\xi_1, \xi_2]_\mathfrak{m}, \xi_3) + \kappa([\xi_1, \xi_3]_\mathfrak{m}, \xi_2) + + \kappa([\xi_3, \xi_2]_\mathfrak{m}, \xi_1)\Big)$$

Since

$$\kappa([\xi_1, \xi_2]_\mathfrak{m}, \xi_3) = -\mathrm{tr}\,({}^t\alpha_3\alpha_1 B_2) + \mathrm{tr}\,({}^t\alpha_3\beta_1 A_2) + \mathrm{tr}\,({}^t\alpha_3\alpha_2 B_1) - \mathrm{tr}\,({}^t\alpha_2\beta_2 A_1)$$

$$+\mathrm{tr}\,({}^t\beta_{\,3}\alpha_1 A_2) + \mathrm{tr}\,({}^t\beta_{\,3}\beta_1 B_2) - \mathrm{tr}\,({}^t\beta_{\,3}\alpha_2 A_1) - \mathrm{tr}\,({}^t\beta_{\,3}\beta_2 B_1)$$

(and analogously for the remaining terms of $(\star)$), the sum on the right hand side of $(\star)$ vanishes.

QED

Theorem 4.4 *Let Γ be an arbitrary discrete subgroup of G_ω. Then the action of Γ on $\mathcal{Z}_\omega$ is properly discontinuous.*

PROOF: We have to prove that if K_1 and K_2 are compact sets in $\mathcal{Z}_\omega$, then the set

$$\Gamma_{K_1,K_2} := \{\gamma \in \Gamma \,|\, \gamma \cdot K_1 \cap K_2 \neq \emptyset\}$$

is finite. First we prove that Γ_{K_1,K_2} is compact. Let $\gamma \in \Gamma_{K_1,K_2}$ and $P_\gamma \in \gamma \cdot K_1 \cap K_2$. Define $P'_\gamma := \gamma^{-1} \cdot P_\gamma = \gamma^{-1} P_\gamma \gamma$. Both P_γ and P'_γ lie in a compact set, and have the following form

$$P_\gamma = \begin{pmatrix} J & 0 \\ L & I \end{pmatrix} \qquad\qquad P'_\gamma = \begin{pmatrix} J' & 0 \\ L' & I \end{pmatrix}$$

Now, ${}^tJ\,\omega$ is a symmetric positive definite matrix, and also ${}^tJ\,'\omega$ is. Indeed,

$$\omega = \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}$$

where Δ =diag$(d_1, ..., d_n)$ being d_i positive integer numbers (cf. §3). Moreover, since $J \in Sp^\omega_{2n}(\mathbb{R})$ and $J^2 = -I$, we can write $J = AJ_0A^{-1}$, where $A \in Sp^\omega_{2n}(\mathbb{R})$), so that

$${}^t({}^tJ\,\omega) = {}^t\omega\, J = -\omega J = {}^tJ\,\omega$$

and

$${}^tJ\,\omega = -{}^tA^{\,-1} J_0 {}^tA\,\omega = -{}^tA^{\,-1} J_0 \omega A^{-1}$$

Since $J_0\omega$ is negative definite, ${}^tJ\,\omega$ is positive, so there exists an orthogonal matrix Q such that ${}^tJ\,\omega = {}^tQ\,DQ$ where D is diagonal and positive definite.

Let

$$\gamma = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}$$

By definition

$$(*) \qquad P'_\gamma = \begin{pmatrix} J' & 0 \\ L' & I \end{pmatrix} = \begin{pmatrix} A^{-1}JA & 0 \\ CA^{-1}(I-J)A + LA & I \end{pmatrix}$$

This equality implies $J' = A^{-1}JA$, hence ${}^tJ'\omega = {}^tA\,{}^tJ\,\omega A$, so that $Q'D'{}^tQ' = {}^tA\,{}^tQ\,DQA$, and consequently

$$D' = {}^tS\,DS$$

Since $S = QAQ'$ varies in a compact set, it follows that A lies in a compact set, as well.

Since, by (*)

$$C = \frac{1}{2}(L' - LA)A^{-1}(I + J)A$$

and, both L and L' are in a compact set, C belongs to a compact set.

But a compact set in a discrete space is finite. This ends the proof.

QED

In our situation, $\Gamma = G_\omega \cap SL_{2n+k}(\mathbb{Z})$. Thank to the previous theorem, and in view of the well known result of Henri Cartan (cf. [4]), that we quote hereafter

Theorem *Let X be an analytic space and Γ a group acting on X in a proper discontinuous way by automorphisms of X. Then the quotient space X/Γ is endowed with a structure of analytic space. Moreover, if X is normal then X/Γ is normal too.*

We finally obtain the following

Theorem 4.5 *The moduli space $\mathcal{M}^\omega_{n,k}$ of polarized foliated tori is a normal analytic space.*

Remark 4.6 *In the special case $n = 1$, i.e. for a general complex foliated torus of the form $\mathbb{C} \times \mathbb{R}^k/\Lambda$ the spaces $\mathcal{Z}$ and $\mathcal{Z}_\omega$ are the same. So the same proof of the previous theorem applies.*

Remark 4.7 *As the referee pointed out to us, since the action of Γ is properly discontinuous on the complex manifold $\mathcal{Z}_\omega$, the space $\mathcal{M}^\omega_{n,k}$ is actually a complex orbifold.*

Of course, in the case of complex foliated tori (not necessarily polarized) $\mathbb{C} \times \mathbb{R}^k/\Lambda$, the moduli space shares the same property as in the polarized foliated case (cf. Remark 3.5).

5 The moduli space $\mathcal{M}^{\omega}_{1,1}$

In this section we want to describe explicitly the simplest moduli space of polarized complex tori: $\mathcal{M}^{\omega}_{1,1}$. Let us give a geometric description of this space, by using the same notation as §2. As we remarked there, $\mathcal{P} = \mathbb{CP}^2 \setminus \mathbb{RP}^2$.

Now the kernel of the polarization ω is a line in $\mathbb{R}^3$, hence a point $p_0 \in \mathbb{RP}^2$. Consider the projection

$$\begin{aligned} \sigma : \mathbb{CP}^2 \setminus \mathbb{RP}^2 &\to \mathbb{RP}^{2\check{}} \\ p &\mapsto \Gamma_p \end{aligned}$$

(where Γ_p is the unique real projective line in $\mathbb{RP}^2$ whose complexification contains p), and the set $\{\Gamma \in (\mathbb{RP}^2)\check{}\ |\, p_0 \in \Gamma\} \cong \mathbb{RP}^1$. Then

$$U = \sigma^{-1}((\mathbb{RP}^2)\check{} \setminus \mathbb{RP}^1) \subset \mathbb{CP}^2 \setminus \mathbb{RP}^2$$

Next consider the set (isomorphic to a copy of $\mathbb{CP}^1$) of all the complex lines in $\mathbb{CP}^2$ passing through p_0, and the canonical projection $\Pi : \mathbb{CP}^2 \setminus \{p_0\} \to \mathbb{CP}^1$ onto it, and restrict it to U:

$$\Pi|_U : U \to \mathbb{CP}^1 \setminus \mathbb{RP}^1$$

the $\mathbb{RP}^1$ being the set of real lines whose complexified pass through p_0.

This map $\Pi|_U$ defines a trivial fibration, with standard fiber $\mathbb{C}$, thus $U = (\mathcal{H}^+ \cup \mathcal{H}^-) \times \mathbb{C}$ (where $\mathcal{H}^{\pm}$ are copies of the Poincaré half-plane).

The group G_ω (of all 3×3 matrices which preserve both the form ω and its kernel) acts on $\mathbb{CP}^2 \setminus \mathbb{RP}^2$, preserving the fibration $\Pi|_U$, and hence the space U too. Therefore our moduli space fits into the commutative diagram:

$$\begin{tikzcd} U \arrow[r, "\Pi|_U"] \arrow[d] & \mathbb{CP}^1 \setminus \mathbb{RP}^1 \arrow[d] \\ \mathcal{M}^{\omega}_{1,1} \arrow[r] & \mathcal{M}_1 \end{tikzcd}$$

where $\mathcal{M}_1$ denotes the moduli space of elliptic curves.

On the other hand, observe that, by Remark 3.6, we may assume that the kernel of ω corresponds to $[0 : 0 : 1] \in \mathbb{RP}^2$, so that we have a natural projection $\pi : \mathbb{CP}^2 \setminus \mathbb{RP}^2 \to \mathbb{CP}^1$ given by $\pi[z_0 : z_1 : z_2] = [z_0 : z_1]$, and that $\mathcal{H}^+ \subset \mathbb{CP}^1 \setminus \mathbb{RP}^1$. Therefore

$$\mathcal{M}^{\omega}_{1,1} = \pi^{-1}(\mathcal{H}^+)/\Gamma$$

Γ being the discrete group of matrices of the form

$$\begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix}$$

where $A \in \mathrm{Sp}_2(\mathbb{Z})$, and the action of Γ on $\mathbb{CP}^2$ is the standard one.

Finally, the Kähler form on $\mathcal{M}^{\omega}_{1,1}$ is the pull back, via π, of the Kähler form on $\mathcal{H}^+$.

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