Examples of Poisson Modules, I

Paolo Caressa

2000

Abstract. We sketch some differential calculus on Poisson algebras and introduce a concept of module and representation on a Poisson algebras; we give examples and consider cohomologies connecting these constructions to the algebra of Poisson brackets.

1 Introduction

In this note we deal with a notion of a module over a Poisson algebra, dwelling mainly on examples. The concept of a Poisson algebra is classical, and the main examples are all of geometric nature (cf. [4]), thus coming from symplectic and, more generally, Poisson manifolds ([1], [6]), and this is also the reason why commutative Poisson algebras are considered, and why the main theme in Poisson algebras is the development of tools which resembles the ones used in geometry, like differential calculus on polyvector fields (cf. [2]). In this note we sketch the already known results and we develop more differential calculus by introducing a concept of connection on Poisson algebras: to do that we also introduce a notion of module over a Poisson algebra, which seems to be very natural and which captures many examples coming from the geometric interpretation of Poisson algebras. To understand the interplay between modules, connections and differential calculus is the aim of this note.

The paper is organised as follows: in the first section we remind the basic definitions on commutative Poisson algebras, which usually are introduced in a geometric way on manifold, and give some example. In the second section we set up an algebraic framework for Cartan and Ricci calculus over associative algebras, through the notion of differential module, and apply it to the case of Poisson algebras. In section three we introduce a notion of Poisson module and give several examples. In section four we exploit the category of Poisson modules, in section five we introduce a notion of representation for a Poisson algebra connected to the concept of Poisson module, while in section six we discuss the relationship between the notions of Poisson module and representation and the concept of connection as introduced in section two.

Acknowledgements. I wish to thank Prof. Paolo de Bartolomeis for helpful advice during the preparation of this paper.

2 Poisson algebras

Remind the following definition:

Definition 2.1 *A Poisson algebra is a $\mathbb{K}$-module A which is both an associative algebra $(A, \cdot)$ and a Lie algebra $(A, \{,\})$ such that the following Leibniz identity holds for each $a, b, c \in A$*

$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$$

($\mathbb{K}$ is a commutative ring with unit).

For us the ground ring will always be a field, and the reader may think about it as the field of real or complex numbers.

So the axioms for a Poisson algebra are the following:

(1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
(2) $\{a, b\} + \{b, a\} = 0.$
(3) $\{\{a, b\}, c\} + \{\{c, a\}, b\} + \{\{b, c\}, a\} = 0.$
(4) $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b.$

Although a Poisson algebra may well be non-commutative (w.r.t. the associative product), our main characters here are commutative[1] ones, thus in algebras such that

$$\forall a, b \in A \quad a \cdot b = b \cdot a$$

Hence from now on the term Poisson algebra *will mean a commutative Poisson algebra.*

[1]We have in mind essentially Poisson algebras of functions, so we are interested in the commutative case; moreover the simplest properties of the derivation functor break down in the non-commutative case.

Examples are well known (since long time ago), the most important being $A = C^\infty(S)$ the algebra of smooth functions[2] on a symplectic manifold (which goes back to Lagrange and Poisson); the main non symplectic example is the Lie–Poisson structure, the Poisson structure on the algebra $C^\infty(\mathfrak{g}^*)$ of smooth functions on the dual vector space of a Lie algebra $\mathfrak{g}$ (and which is due to Lie); more generally, the algebra of functions $C^\infty(M)$ of a Poisson manifold is, by definition, a Poisson algebra (for instance see [2], [4] for these examples and much more).

Poisson algebras are the objects of a category whose morphisms are Poisson maps, thus $\mathbb{K}$-linear maps $f : A \longrightarrow B$ such that

$$\forall a, b \in A \quad f\{a, b\} = \{f(a), f(b)\} \qquad \text{and} \qquad f(ab) = f(a)f(b)$$

Notice that this category has tensor products, since, if A and B are Poisson algebras then $A \otimes B$ becomes in turn a Poisson algebra by means of the following operations:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (b_1 b_2)$$

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} = \{a_1, b_1\} \otimes a_2 b_2 + a_1 b_1 \otimes \{a_2, b_2\}$$

Of course a Poisson subalgebra B of a Poisson algebra A is an associative subalgebra closed under Poisson brackets, and a Poisson ideal is an associative ideal which is also a Lie ideal w.r.t. Poisson brackets.

The most important subalgebra of a given Poisson algebra A is $\operatorname{Cas} A$, the *Casimir subalgebra* which is simply the center of the Lie algebra $(A, \{,\})$:

$$\operatorname{Cas} A = \{c \in A \,|\, \forall a \in A \quad \{a, c\} = 0\}$$

This is not a Poisson ideal.

Because of the Leibniz identity, Poisson brackets induce derivations on the associative algebra $(A, \cdot)$: if we denote by $\operatorname{Der} A$ the A-module (remember: we confine ourselves to commutative algebras) of derivation of A in itself then we have a $\mathbb{K}$-linear map

$$X : A \longrightarrow \operatorname{Der} A$$

defined as

$$X_a(b) = \{a, b\}$$

(so that it is actually a derivation and not simply a linear operator because of the Leibniz identity); Jacobi identity for $\{,\}$ means that X is a Lie algebra morphism:

$$X_{\{a,b\}} = [X_a, X_b]$$

[2]Of course one may consider analytical or simply polynomial functions, both in this and in the following example.

Classically one defines the derivations X_a to be the "Hamiltonian fields" on the algebra A: we denote the Lie algebra of Hamiltonian derivations by $\mathrm{Ham}\, A$: it is a Lie subalgebra of $\mathrm{Der}\, A$. We have the exact sequence of Lie algebras:

$$0 \longrightarrow \mathrm{Cas}\, A \longrightarrow A \longrightarrow \mathrm{Ham}\, A \longrightarrow 0$$

Notice that a Hamiltonian derivation is also a derivation w.r.t. the Lie structure of A, since

$$X_a\{b,c\} = \{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\} = \{X_a b, c\} + \{b, X_a, c\}$$

Then it is natural to consider the set of derivations of A which are also Lie derivations: we denote it by $\mathrm{Can}\, A$ and call its elements *canonical derivations*. Of course it is a Lie subalgebra of $\mathrm{Der}\, A$ and $\mathrm{Ham}\, A$ is a Lie ideal in $\mathrm{Can}\, A$.

The Lie algebra $H^1_\pi(A) = \mathrm{Can}\,(A)/\mathrm{Ham}\,(A)$ (we use this notation because it turns out that this space is actually the first Poisson cohomology space of A) is an important invariant of the Poisson algebra A: for example there is a natural map

$$H^1_\pi(A) \longrightarrow \mathrm{Der}\, \mathrm{Cas}\, A$$

defined as follows: if $X \in \mathrm{Can}\, A$ then $X \mathrm{Cas}\, A \subset \mathrm{Cas}\, A$, since if $c \in \mathrm{Cas}\, A$ then

$$\{X(c), a\} = X\{c,a\} - \{c, X(a)\} = 0$$

for each $a \in A$; of course if $X \in \mathrm{Ham}\, A$ then $Xc = 0$ for each $c \in \mathrm{Cas}\, A$, so that a class $X + \mathrm{Ham}\, A$ defines a derivation $\overline{X}$ in $\mathrm{Cas}\, A$; this map is surjective, since if $Y \in \mathrm{Der}\, \mathrm{Cas}\, A$ then we can use the exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Cas}\, A \longrightarrow A \longrightarrow \mathrm{Ham}\, A \longrightarrow 0$$

to extend Y to a derivation of A (modulo $\mathrm{Ham}\,(A)$); but of course this map is not injective.

The category of Poisson algebras has, of course, a "geometric" dual. Be A a Poisson algebra: then we can consider its spectrum, thus the set $\mathrm{Spec}\, A$ of maximal ideals; if A is commutative we can repeat the usual arguments of Algebraic Geometry and Functional Analysis to give to $\mathrm{Spec}\, A$ some topology. It suffices to consider elements of A as "points" $\chi \in \mathrm{Spec}\,(A)$ in the usual way $a(\chi) = \chi(a)$ (we identify maximal ideals and multiplicative functionals on the algebra). So we can consider the weak topology w.r.t. these functions on A.

Example 2.2 *If $A = C^\infty(M)$ where M is a smooth manifold then of course, as a set,* $\mathrm{Spec}\,(A) = M$. *Moreover our topology coïncides in this case with the manifold topology since a set is closed if and only if it is the zero level set of a smooth function (Whitney theorem).*

Example 2.3 *If $A = C(X)$ (complex continuous functions on a Hausdorff space) then* $\mathrm{Spec}\,(A)$ *is homeomorphic to X, as follows from Gel'fand–Naimark theory.*

Now consider the algebra $\mathrm{Cas}\,A$ of Casimir elements of some Poisson algebra A, and its spectrum $\mathrm{Spec}\,\mathrm{Cas}\,A$ with its topology. Obviously there exists a surjection

$$\Pi : \mathrm{Spec}\,A \longrightarrow \mathrm{Spec}\,\mathrm{Cas}\,A \longrightarrow 0$$

corresponding to the injection $\mathrm{Cas}\,A \subset A$: thus, in some sense, the topological space $\mathrm{Spec}\,A$ defines a fibration on the space $\mathrm{Spec}\,\mathrm{Cas}\,A$.

Theorem 2.4 *Fibers of the map Π are spectra of symplectic Poisson algebras.*

PROOF: Take $\mathfrak{m} \in \mathrm{Spec}\,\mathrm{Cas}\,A$ and $\Pi^{-1}(\mathfrak{m})$: it is the set of maximal ideals which contain the ideal $\mathfrak{m}$. Now, for each $\mathfrak{M} \in \Pi^{-1}(\mathfrak{m})$, consider the quotient $A_{\mathfrak{M}} = \mathfrak{M}/\mathfrak{m}$: it is an associative algebra which is Poisson w.r.t the following brackets:

$$\{a + \mathfrak{m}, b + \mathfrak{m}\} = \{a, b\} + \mathfrak{m}$$

(where $a, b \in \mathfrak{M}$). This definition makes sense because $\mathfrak{m} \subset \mathrm{Cas}\,A$, and these brackets are really Poisson since $\{\ \}$ on A are; now compute Casimir elements for these brackets: if $c + \mathfrak{m}$ is such an element then, for each $a \in \mathfrak{M}$:

$$\{a + \mathfrak{m}, c + \mathfrak{m}\} = \{a, c\} + \mathfrak{m}$$

must belong to $\mathfrak{m}$, which means that $c+\mathfrak{m}$ defines an element in $\mathrm{Cas}\,A/\mathfrak{m} \cong \mathbb{K}$, therefore c is a constant. Hence brackets defined on $A_{\mathfrak{M}}$ are symplectic.

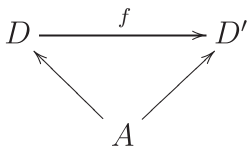
QED

3 A general setting for Differential Calculus

The pair $(\mathrm{Ham}\,A, X)$ plays the role of the pair (Ω_A, d) in classical differential calculus, even thought the set $\mathrm{Ham}\,A$ is not an A-module nor it has the universal property of differentials: so we are forced to define some generalised differential concept in our more general context, and we start with a

Definition 3.1 *A differential module over an associative algebra A is a pair (D, δ), where D is an A-module and $\delta \in \mathrm{Der}\,(A, D)$, such that the image $\mathrm{Im}\,\delta$ spans D as a module.*

Of course differential modules form a category where a morphism between (D,δ) and (D',δ') is an A-module morphism $f : D \longrightarrow D'$ such that the following diagram commutes:



For instance the module Ω_A of Kähler differentials over A defines a differential module (w.r.t. the universal derivation $d : A \longrightarrow \Omega_A$), which can in fact be defined as the initial object in the category of differential modules: then, as usual, one can construct Ω_A explicitly and show that it satisfies the universal property of initial objects; in the sequel we will fix a category of differential modules and denote by Ω_A its initial object[3].

Our example here is the module $\mathcal{H}_A$ generated by $\operatorname{Ham} A$: notice that if (and only if) the Poisson structure is symplectic (i.e. non degenerate: $\operatorname{Cas} A = \mathbb{K}$) then $\mathcal{H}_A = \operatorname{Der} A$ is precisely the dual of Ω_A (Kähler differentials); Leibniz identity means that $(\mathcal{H}_A, X)$ is a differential module. Notice that, by definition of Ω_A as initial object in the category of differential modules, there exists a map

$$\mathbf{H} : \Omega_A \longrightarrow \mathcal{H}_A$$

of A-differential modules, thus A-linear and such that

$$X_a = \mathbf{H}(da)$$

so that we can define Poisson brackets as

$$\{a, b\} = \langle \mathbf{H}(da), db \rangle$$

which we may rewrite as

$$\{a, b\} = \pi(da, db)$$

where $\pi : \Omega_A \wedge \Omega_A \longrightarrow A$ is the tensor determined by $\mathbf{H}$, which is nothing else than the *Poisson tensor* of the algebra, and which indeed characterises the Poisson structure by means of the well known integrability condition $[\![\pi, \pi]\!] = 0$, being $[\![,]\!]$ the Schouten–Nijenhuis brackets on polyderivations,

[3] Notice that we have to assume that our category of differential modules has an initial object: if we confine ourselves to the category of *all* differential modules then Kähler differentials do the job, and if we consider projective differential modules over $A = C^\infty(M)$, the algebra of smooth maps on a differential manifold, then the initial object is the module of de Rham differential, which is distinct from that of Kähler differentials.

cf. e.g. [2] (the map **H** characterises the Poisson structure too, as showed in [5] where Schouten brackets are introduced for such operators).

We can develop classical Cartan calculus on differential modules (D, δ), by defining a contraction map and a Lie derivative: to do this we have to consider the following submodule of $\mathrm{Der}\, A$

$$\mathfrak{X}(D) = \{X \in \mathrm{Der}\, A \,|\, \forall c \in \ker\delta \quad X(c) = 0\}$$

This is a submodule and a Lie subalgebra too; it is just the space of derivations which see the elements of the kernel $\ker\delta$ as constants. Next we define a map $\mathbf{i} : \mathfrak{X}(D) \times D \longrightarrow A$ as

$$\mathbf{i}_X \delta a = X(a)$$

and extend by A_δ-linearity. This map is called *contraction* and it is a non degenerate pairing which satisfies the usual properties.

We can define also a Lie derivative by taking Cartan's "magic formula" as a definition

$$\mathcal{L}_X\omega = \mathbf{i}_X\delta\omega + \delta\mathbf{i}_X\omega$$

for $X \in \mathfrak{X}(D)$ and $\omega \in D$. Then, by extending these maps to the exterior powers of the module D respecting degrees, we find that *mutatis mutandis* all the usual identities of differential calculus hold (for example those listed in the tables in [1, page 121] or in [6, pag. 126–128]).

We can extend this calculus to higher order "differentials" by considering the spaces[4] $\bigwedge_A^n D$ and extending the derivation $\delta : A \longrightarrow D$ to a sequence of maps $\delta : \bigwedge^n D \longrightarrow \bigwedge^{n+1} D$ as

$$\delta(a_0\delta a_1 \wedge \cdots \wedge \delta a_n) = \delta a_0 \wedge \delta a_1 \wedge \cdots \wedge \delta a_n$$

Both contraction and Lie derivative extends to higher order preserving usual properties, moreover $\delta \circ \delta = 0$: thus we can consider the *differential cohomology* of A w.r.t. the differential module (D, δ):

$$H_D(A) = \ker\delta / \mathrm{Im}\,\delta$$

The subalgebra $\ker\delta$ contains informations about how much a differential module is not an initial object in its category: indeed consider the algebra $A_\delta = A \otimes_{\mathbb{K}} \ker\delta$ over the ring $\ker\delta$:

Proposition 3.2 $\Omega_{A_\delta} = D$ *and* $\mathrm{Der}\, A_\delta = \mathfrak{X}(D)$.

[4] In a non commutative setting tensor product should be taken into account instead of wedge one.

In fact there is a map $\Xi : \mathfrak{X}(D) \longrightarrow \mathrm{Der}\, A_\delta$ given by

$$\Xi(X)(a \otimes c) = X(a) \otimes c$$

which is an isomorphism; hence Kähler differentials over A_δ are characterised as

$$\mathfrak{X}(D) = \mathrm{Der}\, A_\delta = \mathrm{Hom}\,_A(\Omega_{A_\delta}, A)$$

so that $\Omega_{A_\delta} = D$.

We can also generalise Ricci calculus to differential modules in the obvious way:

Definition 3.3 *If (D, δ) is an A-differential module and E is an A-module, a D-connection in E is a $\mathbb{K}$-linear map $\nabla : E \longrightarrow E \otimes D$ such that (for $a \in A$ and $e \in E$)*

$$\nabla(ae) = a\nabla e + e \otimes \delta a$$

Of course a connection is $\ker \delta$-linear; for instance, if $D = \Omega_A$ (the initial object in the category of modules we are dealing with) then we recover the usual concept of a connection, and the following theorem, due to Nahrasiman, is well known (cf. e.g. [3]):

Theorem 3.4 *An A-module E has a Ω_A-connection if and only if is A-projective.*

In our more general context this will not be the case; we have to slightly generalise this result as follows: if E is an A-module it is also an A_δ-module via the action $(a \otimes c) \cdot e = (ac) \cdot e$.

Theorem 3.5 *An A-module E has a D-connection if and only if is A_δ-projective.*

The proof is the same as given in [3].

Of course such an operator extends to the exterior powers $E \otimes \bigwedge^k D$ as

$$\nabla(e \otimes \omega) = \nabla e \wedge \omega + (-1)^{\deg \omega} e \wedge \delta\omega$$

so that the curvature $R = \nabla^2$ is well defined and satisfies Bianchi identity:

$$\nabla R = 0$$

We can also reformulate the concept of D-connection in terms of "partial" covariant derivatives as follows

Definition 3.6 *If (D, δ) is an A-differential module and E an A-module then a D-covariant derivative in E is a $\mathbb{K}$-bilinear map $\mathbf{D} : \mathfrak{X}(D) \times E \longrightarrow E$ such that*

$$\mathbf{D}_X(ae) = a\mathbf{D}_X e + \delta(a)e$$

(if $X \in \mathfrak{X}(D)$, $a \in A$ and $e \in E$.)

Of course, if $D = \Omega_A$ then $\mathfrak{X}(D) = \mathrm{Der}\, A$ and we recover the classical concept of covariant derivative.

A D-covariant derivative induces a connection as follows: simply define

$$\mathbf{i}_X \nabla e = \mathbf{D}_X e$$

for $X \in \mathfrak{X}(D)$ and $e \in E$; because of the non-degeneracy of the contraction between $\mathfrak{X}(D)$ and D this equation uniquely defines a map $\nabla : E \longrightarrow E \otimes D$, which is of course a connection:

$$\mathbf{i}_X \nabla(ae) = \mathbf{D}_X(ae) = a\mathbf{D}_X e + (\mathbf{i}_X \delta a)\, e = \mathbf{i}_X\, (a\nabla e + e \otimes \delta a)$$

Needless to say, the curvature of the connection is the obstruction of the covariant derivative to be a morphism of Lie algebras:

$$\mathbf{i}_X \mathbf{i}_Y R = \mathbf{D}_X \mathbf{D}_Y - \mathbf{D}_Y \mathbf{D}_X - \mathbf{D}_{[X,Y]}$$

For instance if A is a Poisson algebra and $D = \mathcal{H}_A$ we have the concept of *Hamiltonian connection*, i.e. a $\mathbb{K}$-linear map $\nabla : E \longrightarrow E \otimes \mathcal{H}_A$ such that

$$\nabla(ae) = a\nabla e + e \otimes X_a$$

To identify the corresponding partial covariant derivative we have to know what $\mathfrak{X}(D)$ is: since there exists the exact sequence

$$0 \longrightarrow \mathrm{Cas}\, A \longrightarrow A \longrightarrow \mathrm{Ham}\, A \longrightarrow 0$$

and $\mathrm{Cas}\, A = \ker \delta$, then $\mathfrak{X}(D) = \mathcal{H}_A = D$. Therefore a Hamiltonian covariant derivative is a $\mathbb{K}$-bilinear map $\mathbf{D} : \mathcal{H}_A \times E \longrightarrow E$ such that

$$\mathbf{i}_{X_a} \nabla e = \mathbf{D}_{X_a} e$$

Leibniz identity reads now as

$$\mathbf{D}_{X_a}(be) = b\mathbf{D}_{X_a} e + \{a, b\}e$$

We recognise in our Hamiltonian connections the *contravariant connections* as defined by Vaisman in [7] who was, in turn, inspired by Bott's works on characteristic classes of foliations.

4 Modules over Poisson algebras

Now we come back to Poisson algebras: the most powerful idea to understand the structure of an algebraic object is to look for its "incarnations", which usually define a category: so for a group we have the category of its representations, for a ring the category of its modules, &c. It is therefore natural, when A is a Poisson algebra, to try to extend Poisson brackets to a suitable category of modules over A. We propose the following

Definition 4.1 *A Poisson module over A is an A-module E endowed with a $\mathbb{K}$-linear map $\lambda : A \times E \longrightarrow E$ such that*

$$\lambda(\{a,b\},e) = \lambda(a,\lambda(b,e)) - \lambda(b,\lambda(a,e))$$

$$\{a,b\} \cdot e = a \cdot \lambda(b,e) - \lambda(b, a \cdot e)$$

for each $a,b \in A$ and $e \in E$ (and $\cdot$ denotes the associative module action).

In other words, a Poisson module is a module both for the associative and for the Lie structure on A, and satisfies some kind of Leibniz rule. It is natural (and useful to control the length of formulas) to avoid any explicit mention of λ and to write $\{a,e\} = \lambda(a,e)$ so that the axioms for a Poisson module become

$$\{\{a,b\},e\} = \{a,\{b,e\}\} - \{b,\{a,e\}\}$$
$$\{a,b\} \cdot e = a \cdot \{b,e\} - \{b, a \cdot e\}$$

Notice that the structure of associative and Lie module on A do not commute in general (of course they do on the Casimir subalgebra $\operatorname{Cas} A$).

Let us collect some example.

Example 4.2 *A is a Poisson module w.r.t. the adjoint actions on itself; also the dual $\mathbb{K}$-vector space A' is a Poisson module w.r.t. the coadjoint actions. In the former case Poisson module axioms coincïde with Poisson algebra axioms; in the latter it is a matter of a simple computation: if $\varphi \in A'$ and $a,b,c \in A$:*

$$\begin{aligned} (\{a,b\}\varphi)(c) &= \varphi(\{a,b\}c) = \varphi(\{a,bc\}) - \varphi(b\{a,c\}) \\ &= \{a,\varphi\}(bc) - (b\varphi)(\{a,c\}) = (b\{a,\varphi\} - \{a,b\varphi\})(c) \end{aligned}$$

Example 4.3 *A Poisson ideal $I \mathrm{p} A$ is a Poisson module w.r.t. the Poisson operations of A restricted to I.*

Example 4.4 *If $\varphi : A \longrightarrow B$ is a morphism of Poisson algebras then B is a Poisson A-module via*

$$a \cdot b = \varphi(a)b \qquad \text{and} \qquad \{a, b\} = \{\varphi(a), b\}$$

That B is a representation of the Lie algebra A is well-known; moreover

$$\begin{aligned} \{a, a'\} \cdot b &= \varphi(\{a, a'\})b = \{\varphi(a), \varphi(a')\}b \\ &= \{\varphi(a), \varphi(a')b\} - \varphi(a')\{\varphi(a), b\} \\ &= \{a, a'b\} - a'\{a, b\} \end{aligned}$$

For instance

Example 4.5 *If A is a non-commutative Poisson algebra, and $Z(A)$ its Poisson center, thus the subalgebra*

$$Z(A) = \{z \in A \,|\, \forall a \in A \quad az = za\}$$

then A is a Poisson module over the Poisson algebra $Z(A)$, w.r.t. the product and Poisson bracket in A.

Example 4.6 *Consider the space of linear operators $\mathrm{End}_{\mathbb{K}}(A)$ on A as an A-module via ($a, b \in A$, $T \in \mathrm{End}_{\mathbb{K}}(A)$):*

$$(aT)(b) = a(Tb)$$

Furthermore we can define

$$\{a, T\}(b) = \{a, Tb\}$$

That this is a Lie action follows from Jacobi identity for the Poisson structure on A; $\mathrm{End}_{\mathbb{K}}(A)$ becomes, equipped with these brackets, a Poisson module on A: indeed

$$(\{a, b\}T)(c) = \{a, b\}T(c) = \{a, bT(c)\} - b\{a, T(c)\} = (\{a, bT\} - b\{a, T\})(c)$$

This is the adjoint Poisson structure: the coadjoint structure

$$\{a, T\}' = -T \circ X_a$$

defines a Lie action too, but it is not Poisson; however, if we consider this Lie structure and the coadjoint associative product:

$$(a \cdot' T)(b) = T(ab)$$

we get a Poisson module.

Notice that there exists a third Lie action which is natural to consider on $\mathrm{End}_{\mathbb{K}}(A)$, namely the difference between the previous ones:

$$\{a, T\}'' = \{a, T\} - \{a, T\}' = [X_a, T]$$

In fact this defines both a Lie representation

$$\begin{aligned}\{\{a, b\}, T\}'' &= [X_{\{a,b\}}, T] = [[X_a, X_b], T] = [X_a, [X_b, T]] - [X_b, [X_a, T]] \\ &= \{a, \{b, T\}''\}'' - \{b, \{a, T\}''\}''\end{aligned}$$

and a Poisson action

$$\{a, bT\}'' = [X_a, bT] = b[X_a, T] + \{a, b\}T = b\{a, T\}'' + \{a, b\}T$$

so that $\{\,\}''$ makes of $\mathrm{End}_{\mathbb{K}}(A)$ a Poisson module.

Example 4.7 *Consider the module* $\mathrm{Der}\, A$ *of derivations on the associative algebra* A*: it is a submodule of* $\mathrm{End}_{\mathbb{K}}(A)$*, but only one of the three Lie actions we defined on* $\mathrm{End}_{\mathbb{K}}(A)$ *sends derivations in derivations: the latter, which now we write as*

$$\{a, X\} = [X_a, X]$$

and induces on $\mathrm{Der}\, A$ *a Poisson structure.*

The module $\mathrm{Der}\, A$ has many interesting submodules: the most important for us is the module $\mathcal{H}_A$ generated by Hamiltonian derivations which is of course a submodule, since

$$[X_a, \sum_i b_i X_{h_i}] = \sum_i (b_i[X_a, X_{h_i}] + \{a, b_i\}X_{h_i}) = \sum_i \big(b_i X_{\{a,h_i\}} + \{a, b_i\}X_{h_i}\big)$$

Example 4.8 *Be* E *and* F *two* A*-modules, and consider the space* $\mathrm{Hom}_{\mathbb{K}}(E, F)$ *of* $\mathbb{K}$*-linear operators* $E \longrightarrow F$*: it is an* A*-module w.r.t. the adjoint action* $aT(e) = a(Te)$*; moreover notice that there's also another natural structure of* A*-module on* $\mathrm{Hom}_{\mathbb{K}}(E, F)$*, namely the coadjoint one:* $aT(e) = T(ae)$*, and that these two structures induce on* $\mathrm{Hom}_{\mathbb{K}}(E, F)$ *a bimodule structure.*

Now consider the subspace $\mathrm{Hom}_A(E, F)$ of A-linear maps: $T \in \mathrm{Hom}_A(E, F)$ if and only if

$$T(ae) = aT(e)$$

($a \in A$ and $e \in E$). In other words, it is the space on which the two actions do coincide.

If F is a Lie module then

$$\{a, T\}_E(e) = \{a, Te\}$$

turns $\mathrm{Hom}_{\mathbb{K}}(E, F)$ into a representation of the Lie algebra A; if F is Poisson then $\mathrm{Hom}_{\mathbb{K}}(E, F)$ is Poisson too. If E is a representation of the Lie algebra A then we can consider the representation on $\mathrm{Hom}_{\mathbb{K}}(E, F)$ given by

$$\{a, T\}_F(e) = T\{a, e\}$$

If E is Poisson w.r.t. the associative coadjoint action then $\mathrm{Hom}_{\mathbb{K}}(E, F)$ is Poisson too.

Hence, in the case of the module $\mathrm{Hom}_A(E, F)$ we have two Lie structures, and their difference

$$\{a, T\} = \{a, T\}_E - \{a, T\}_F$$

The latter induces on $\mathrm{Hom}_A(E, F)$ a structure of Poisson module whenever F is:

$$\begin{aligned}\{a, bT\}(e) &= \{a, bTe\} - bT\{a, e\} = b\{a, Te\} - bT\{a, e\} + \{a, b\}Te \\ &= b\{a, T\}(e) + \{a, b\}T(e)\end{aligned}$$

Example 4.9 *The module Ω_A of differentials is also a Poisson module via the action*

$$\{a, \omega\} = \mathcal{L}_{X_a}\omega$$

Since $\mathcal{L}_{[X_a, X_b]} = [\mathcal{L}_{X_a}, \mathcal{L}_{X_b}]$ these brackets defines a Lie representation, which is a Poisson structure since

$$a\{b, \omega\} - \{b, a\omega\} = a\mathcal{L}_{X_b}\omega - \mathcal{L}_{X_b}a\omega = a\mathcal{L}_{X_b}\omega - \{b, a\}\omega - a\mathcal{L}_{X_b}\omega = \{a, b\}\omega$$

We remark explicitly that this structure of Poisson module is compatible with the Lie brackets on Ω_A induced by the Poisson structure on A and defined as[5]

$$\{\omega_1, \omega_2\} = \mathcal{L}_{\pi^\# \omega_1}\omega_2 - \mathcal{L}_{\pi^\# \omega_2}\omega_1 - d\pi(\omega_1, \omega_2)$$

In fact: $\{da, \omega\} = \mathcal{L}_{X_a}\omega = \{a, \omega\}$.

[5]Remember that the Poisson structure can be defined in terms of the Poisson tensor $\pi : \Omega_A \wedge \Omega_A \longrightarrow A$: we write $\pi^\#$ for the map $\Omega_A \longrightarrow \mathrm{Der}\, A$ such that $\pi^\# da = X_a$, borrowing this notation from Differential Geometry.

Notice that the Leibniz identity we gave in the definition of a Poisson module is not the unique possible: in fact we could ask as well for the following identity to hold:

$$\{ab, e\} = a\{b, e\} + b\{a, e\} \tag{M}$$

This latter identity takes into account the associative product of the Poisson algebra, while the former concerned the Poisson bracket.

For instance if $E = A$ then the (M) identity is obviously satisfied; also for $E = A'$, since

$$\begin{aligned}\{ab, \varphi\}(c) &= \varphi(\{ab, c\}) = \varphi(a\{b, c\}) + \varphi(b\{a, c\}) \\ &= (a\varphi)\{b, c\} + (b\varphi)\{a, c\} \\ &= \{a, b\varphi\}(c) + \{b, a\varphi\}(c)\end{aligned}$$

Notice that

$$\{a, be\} + \{b, a\} = b\{a, e\} - \{a, b\}e + a\{b, e\} - \{b, a\}e = a\{b, e\} + b\{a, e\}$$

and so the relationship between the two Leibniz identities is expressed by the

Lemma 4.10 $\{a, be\} + \{b, ae\} = a\{b, e\} + b\{a, e\}$.

A Poisson module does not necessarily fulfils identity (M): it suffices to take $E = \operatorname{Der} A$:

$$\{ab, D\} = [X_{ab}, D] = [aX_b + bX_a, D] = a[X_b, D] - (Da)X_b + b[X_a, D] - (Db)X_a$$

Definition 4.11 *An A-module E is called multiplicative if it also a Lie module and identity* (M) *holds for each $a, b \in A$ and $e \in E$.*

For example A and A' (w.r.t. the Poisson structure we considered on them), while, as just said, $\operatorname{Der} A$ is not, nor Ω_A is multiplicative, since

$$\{a, \omega\} = a\mathcal{L}_{X_b}\omega + b\mathcal{L}_{X_a}\omega + \omega(X_b)da + \omega(X_a)db$$

$\operatorname{End}_{\mathbb{K}}(A)$ is multiplicative only w.r.t. the Poisson structures we called $\{\,\}$ and $\{\,\}'$, but not w.r.t. the third one.

To complete the picture we give an example of multiplicative module which is not Poisson: be $\mathfrak{g}$ a Lie algebra and $\rho : \mathfrak{g} \longrightarrow \operatorname{End}(V)$ a representation of $\mathfrak{g}$; consider the Lie–Poisson algebra $C^\infty(\mathfrak{g}^*)$ (one could work at a purely algebraic level considering $\mathrm{S}(\mathfrak{g}^*)$ instead) and the space $C^\infty(\mathfrak{g}^*, V)$ of vector valued smooth functions on the linear manifold $\mathfrak{g}^*$; we claim that this is a multiplicative module: here the action is ($f \in C^\infty(\mathfrak{g}^*)$, $\varphi \in C^\infty(\mathfrak{g}^*, V)$ and $x \in \mathfrak{g}^*$)

$$\{f, \varphi\}(x) = \rho(df_x)(\varphi(x))$$

(we consider a covector at a point as an element of the Lie algebra $\mathfrak{g}$: $df_x \in T_x^*\mathfrak{g}^* \cong \mathfrak{g}^{**} \cong \mathfrak{g}$); this defines a representation of the Lie algebra $(C^\infty(\mathfrak{g}^*), \{\,\})$ on $C^\infty(\mathfrak{g}^*, V)$, since

$$\begin{aligned}\{\{f, g\}, \varphi\}(x) &= \rho(d\{f, g\}_x)(\varphi(x)) = \rho([df_x, dg_x])(\varphi(x)) \\ &= \rho(df_x)(\rho(dg_x)(\varphi(x))) - \rho(dg_x)(\rho(df_x)(\varphi(x))) \\ &= \{f, \{g, \varphi\}\}(x) - \{g, \{f, \varphi\}\}(x)\end{aligned}$$

This module is multiplicative:

$$\{fg, \varphi\}(x) = \rho(f(x)dg_x + g(x)df_x)(\varphi(x)) = f(x)\{g, \varphi\}(x) + g(x)\{f, \varphi\}(x)$$

but it is not Poisson:

$$\{f, g\varphi\}(x) = \rho(df_x)(g(x)\varphi(x)) = g(x)\rho(df_x)(\varphi(x)) = g(x)\{f, \varphi\}(x)$$

Of course this example is of geometric nature: if G is a Poisson–Lie group and E a vector bundle whose fibers are representations of the dual group G^* then the module of sections $\Gamma(G, E)$ is multiplicative over the algebra $C^\infty(G)$.

5 Simple constructions on Poisson modules

We want to set up a category of Poisson modules, so we need the concept of a morphism between Poisson modules, but the most obvious one is not the most suitable one: in fact the temptation is to define $f : E \longrightarrow F$ as a Poisson morphism if $f(ae) = af(e)$ and $f(\{a, e\}) = \{a, f(e)\}$, but in this case we would have, when $E = F = A$:

$$f\{a, b\} = f(\{a, b \cdot 1) = \{a, b\}f(1)$$

So we state the

Definition 5.1 *If E and F are Poisson modules over A then a Poisson morphism $f : E \longrightarrow F$ is a* Cas *A-linear map such that for $a \in A$ and $e \in E$:*

$$f\{a, e\} = \{a, f(e)\}$$

For example, the Hamiltonian map $X : A \longrightarrow \mathcal{H}_A$ is a Poisson morphism (w.r.t. the Poisson module structures defined above), since

$$X_{\{a,b\}} = [X_a, X_b] = \{a, X_b\}$$

and $X_{ca} = cX_a$ when $c \in \operatorname{Cas} A$.

Another example of Poisson morphism is $\pi^\# : \Omega_A \longrightarrow \operatorname{Der} A$:

$$\pi^\#\{a,\omega\} = \pi^\#\{da,\omega\} = [\pi^\# da, \pi^\#\omega] = [X_a, \pi^\#\omega] = \{a, \pi^\#\omega\}$$

The space $\operatorname{Hom}_{\mathrm{Pos}}(E,F)$ of Poisson morphisms between two Poisson modules is again a Poisson module w.r.t. the action

$$\{a,\varphi\}(e) = \varphi\{a,e\}$$

and it is a multiplicative module whenever E is.

Notice that, if $E = F = A$ then a Poisson morphism between the two module structure is not a Poisson morphism of algebras.

A more interesting functorial construction is the following: be E and F Poisson modules: then also $E \otimes_A F$ has a natural structure of Poisson module defined as

$$\{a, e\otimes e'\} = \{a,e\}\otimes e' + e\otimes\{a,e'\}$$

Of course this is a representation of the Lie algebra A and moreover

$$\begin{aligned}
\{a, be\otimes e'\} &= \{a,be\}\otimes e' + be\otimes\{a,e'\} \\
&= b\{a,e\}\otimes e' - \{b,a\}e\otimes e' + be\otimes\{a,e'\} \\
&= b\{a,e\otimes e'\} - \{b,a\}e\otimes e'
\end{aligned}$$

If E and F are multiplicatives, also $E \otimes_A F$ is:

$$\begin{aligned}
\{ab, e\otimes e'\} &= a\{b,e\}\otimes e' + b\{a,e\}\otimes e' + e\otimes a\{b,e'\} + e\otimes b\{a,e'\} \\
&= a\{b,e\otimes e'\} + b\{a,e\otimes e'\}
\end{aligned}$$

Now consider a Poisson algebra A and an A-module E which is also a representation of the Lie algebra $(A,\{,\})$: we can define the spaces of derivations w.r.t. associative and Lie module structure on E:

$$\operatorname{Der}(A,E) = \{X \in \operatorname{End}_{\mathbb{K}}(A,E) \,|\, \forall a,b\in A \quad X(ab) = aX(b) + bX(a)\}$$

$$\operatorname{Der}_{\mathrm{Lie}}(A,E) = \{X \in \operatorname{End}_{\mathbb{K}}(A,E) \,|\, \forall a,b\in A \quad X\{a,b\} = \{a,Xb\} - \{b,Xa\}\}$$

and their intersection

$$\operatorname{Can}(A,E) = \operatorname{Der}(A,E) \cap \operatorname{Der}_{\mathrm{Lie}}(A,E)$$

which we call *space of canonical derivations with coefficients in E*; of course we have also *Hamiltonian operators with coefficients in E*, defined as

$$X_e a = -\{a, e\}$$

and they form a subspace $\mathrm{Ham}\,(A, E)$ of $\mathrm{Can}\,(A, E)$.

Proposition 5.2 *An A-module E is multiplicative if and only if there exists a $\mathbb{K}$-linear operator $X : E \longrightarrow \mathrm{Der}\,(A, E)$ such that*

$$X_e\{a, b\} = X_{X_e a} b + X_{X_e b} a$$

and E is Poisson if and only if there exists a $\mathbb{K}$-linear operator $X : E \longrightarrow \mathrm{Der}_{\,\mathrm{Lie}}(A, E)$ such that

$$X_{ae} = aX_e + \{a, -\}e$$

(thus $X_{ae} b = aX_e b + \{a, b\}e$).

Now we go back to some examples of the previous section: we remarked that a Poisson ideal I in a Poisson algebra A is a Poisson module; of course A is an extension with kernel I (in an obvious sense). As usual, suppose I to be an abelian ideal (as Lie algebra: remember that our Poisson algebras are all abelian in the associative sense) in A: then we can consider the quotient Poisson algebra A/I, and I is a Poisson module over A/I too:

$$\{a + I, i\} = \{a, i\}$$

is well defined since I is abelian. On the other hand we can classify extensions with abelian kernel in the following way: start with a Poisson algebra A and a Poisson module E and consider its extensions B:

$$0 \longrightarrow E \longrightarrow B \longrightarrow A \longrightarrow 0$$

As usual to build B one just considers the vector space $A \oplus E$ equipped with the operations

$$(a \oplus e)(a' \oplus e') = aa' \oplus (ae' + a'e)$$

$$\{a \oplus e, a' \oplus e'\} = \{a, a'\} \oplus (\{a, e'\} - \{a', e\})$$

Suppose A to be a Poisson algebra, E an A-module (w.r.t. the associative structure) and define on $A \oplus E$ the two previous operations:

Proposition 5.3 *$A\oplus E$ is a Poisson algebra if and only if E is a multiplicative Poisson module.*

PROOF: Be $A\oplus E$ a Poisson algebra: then define

$$a\cdot e=(a\oplus 0)(0\oplus e)\qquad\text{and}\qquad\{a,e\}=\{a\oplus 0,0\oplus e\}$$

Then axioms for a Poisson algebra imply exactly axioms for Poisson multiplicative structure on E. Vice versa, if E is a multiplicative Poisson module, then $A\oplus E$ (according to the previous definitions of product and bracket) is an associative algebra:

$$\begin{aligned}((a\oplus e)(a'\oplus e'))(a''\oplus e'') &= (aa')a''\oplus((aa')e''+a''(ae'+a'e))\\ &= (a\oplus e)(a'a''\oplus(a'e''+a''e'))\\ &= (a\oplus e)((a'\oplus e')(a''\oplus e''))\end{aligned}$$

(notice that we used the commutativity of A at a crucial step) and it is also a Lie algebra:

$$\begin{aligned}\{\{a\oplus e,a'\oplus e'\},a''\oplus e''\} &= \{\{a,a'\},a''\}\oplus(\{\{a,a'\},e''\}-\{a'',\{a,e'\}-\\ &\qquad -\{a',e\}\})\\ &= \{\{a,a'\},a''\}\oplus(\{\{a,a'\},e''\}-\{a'',\{a,e'\}\}+\\ &\qquad +\{a'',\{a',e\}\})\end{aligned}$$

So the obstruction to the Jacobi identity reduces to the vanishing of

$$\begin{aligned}&\{\{a,a'\},e''\}-\{a'',\{a,e'\}\}+\{a'',\{a',e\}\}+\{\{a',a''\},e\}-\{a,\{a',e''\}\}+\\ &\qquad +\{a,\{a'',e'\}\}+\{\{a'',a\},e'\}-\{a',\{a'',e\}\}+\{a',\{a,e''\}\}\end{aligned}$$

which in fact is zero, since $\{\{a,a'\},e''\}=\{a,\{a',e''\}\}-\{a',\{a,e''\}\}$ and so on, being E a representation of the Lie algebra A.

Finally we come to the Leibniz identity:

$$\begin{aligned}\{(a\oplus e)(a'\oplus e'),a''\oplus e''\} &= \{aa',a''\}\oplus(\{aa',e''\}-\{a'',ae'+a'e\})\\ &= (a\{a',a''\}+a'\{a,a''\})\oplus(a\{a',e''\}+\\ &\qquad +a'\{a,e''\}-\{a'',ae'\}-\{a'',a'e\})\\ &= a\{a',a''\}\oplus(a\{a',e''\}-a\{a'',e'\}-\\ &\qquad -\{a',a''\}e)+a'\{a,a''\}\oplus\\ &\qquad \oplus(a'\{a,e''\}-a'\{a'',e\}-\{a,a''\}e')\\ &= (a\oplus e)\{a'\oplus e',a''\oplus e''\}+\\ &\qquad +(a'\oplus e')\{a\oplus e,a''\oplus e''\}\end{aligned}$$

Notice that we used both Poisson and multiplicative structure on E to get the result.

QED

Of course if E is a multiplicative Poisson module then it is also an ideal in $A\oplus E$, and moreover $\{E,E\}=0$ and $E\cdot E=0$ by definition of $\{,\}$ and $\cdot$ on $A\oplus E$.

6 Cohomology and representations of Poisson algebras

We considered so far two constructions which fit very well into a cohomological framework: the quotient $\mathrm{Can}\,(A,E)/\mathrm{Ham}\,(A,E)$ and the extension of Poisson algebras by means of a multiplicative Poisson module.

And in fact one can consider, given a Poisson module E over a Poisson algebra A, the complex of multilinear skew-symmetric maps $P:A\wedge...\wedge A\longrightarrow E$ with the coboundary operators (of degree $+1$)

$$\begin{aligned}(\delta P)(a_0\wedge a_1\wedge...\wedge a_n) &= \sum_{i=0}^{n}(-1)^i\{a_i,P(a_0\wedge...\wedge\widehat{a_i}\wedge...\wedge a_n)\}\\ &+\sum_{i<j}^{0...n}(-1)^{i+j}P(\{a_i,a_j\}\wedge a_0\wedge...\wedge\widehat{a_i}\wedge...\wedge\widehat{a_j}\wedge...\wedge a_n)\end{aligned}$$

and the cohomology $H^\bullet(A,E)$ of this complex.

Proposition 6.1
(1) $H^0(A,E) = \mathrm{Cas}\, E = \{e \in E \mid \forall a \in A\ \{a,e\} = 0\}$;
(2) $H^1(A,E) = \mathrm{Can}\,(A,E)/\mathrm{Ham}\,(A,E)$;
(3) $H^2(A,E) = \{$*extensions* $0 \longrightarrow E \longrightarrow B \longrightarrow A\}/\{$*trivial extensions*$\}$.

PROOF: (1) is trivial; (2) follows from $\delta P(a \wedge b) = \{a, P(b)\} - \{b, P(a)\} - P\{a,b\}$ which implies $\mathrm{Can}\,(A,E) = Z^1(A,E)$, and $P(a) = \delta Q(a) = \{a, Q\}$ which implies $\mathrm{Ham}\,(A,E) = B^1(A,E)$.

Next we come to (3): it is a standard result for Lie algebras, however we show the explicit computations: an abelian extension $0 \longrightarrow E \longrightarrow B \longrightarrow A$ determines a linear section $L : A \longrightarrow B$ of the projection $B \longrightarrow A$; this is a morphism of Poisson algebras if and only if

$$\{L(a), L(a')\} - L\{a, a'\} = 0 \qquad \text{and} \qquad L(a)L(a') - L(aa') = 0$$

Define a map $P : A \wedge A \longrightarrow B$ as

$$P(a \wedge a') = \{L(a), L(a')\} - L\{a, a'\}$$

(Notice that the action of A on E is given by $\{L(a), e\} = \{a, e\}$ since L is a section, so that $L(a) = a' + e'$.)

Now: P is a cocycle in $Z^2(A,E)$: indeed $P(a \wedge a')$ is in the kernel of the projection $B \longrightarrow A$, thus its image is in E; moreover

$$\begin{aligned}
\delta P(a \wedge a' \wedge a'') &= \{a, \{L(a'), L(a'')\} - L\{a', a''\}\} - \{a', \{L(a), L(a'')\} - \\
&\quad - L\{a, a''\}\} + \{a'', \{L(a), L(a')\} - L\{a, a'\}\} - \\
&\quad - \{L\{a, a'\}, L(a'')\} + L\{\{a, a'\}, a''\} + \\
&\quad + \{L\{a, a''\}, L(a')\} - L\{\{a, a''\}, a'\} + \\
&\quad - \{L\{a', a''\}, L(a)\} + L\{\{a', a''\}, a\} \\
&= \{a, \{L(a'), L(a'')\}\} - \{a', \{L(a), L(a'')\}\} + \\
&\quad + \{a'', \{L(a), L(a')\}\} - \{L\{a, a'\}, L(a'')\} + \\
&\qquad + \{L\{a, a''\}, L(a')\} - \{L\{a', a''\}, L(a)\} \\
&= \{a, \{a', L(a'')\}\} - \{a', \{a, L(a'')\}\} + \{a'', \{a, L(a')\}\} + \\
&\quad -\{\{a, a'\}, L(a'')\} + \{\{a, a''\}, L(a')\} - \{\{a', a''\}, L(a)\} \\
&= 0
\end{aligned}$$

(we used Jacoby identity, $\mathrm{Im}\, L \subset E$ and the fact that $\{a, e\} = \{L(a), e\}$).

Now suppose P to be a couboundary: then

$$P(a \wedge a') = \delta Q(a \wedge a') = \{a, Q(a')\} - \{a', Q(a)\} - Q\{a, a'\}$$

so that $R = L - Q : A \longrightarrow E$ is such that

$$\begin{aligned}
\{R(a), R(a')\} &= \{L(a), L(a')\} - \{L(a), Q(a')\} - \{Q(a), L(a')\}+ \\
&\quad + \{Q(a), Q(a')\} \\
&= P(a \wedge a') + L\{a, a'\} - \{a, Q(a')\} + \{a', Q(a)\} + \\
&\quad + \{Q(a), Q(a')\} \\
&= L\{a, a'\} - Q\{a, a'\} = R\{a, a'\}
\end{aligned}$$

(remember that $Q(a) \in E$ which is an abelian ideal).

So we find that P is a coboundary if and only if there's a Lie algebra morphism $A \longrightarrow B$ which is a section of the projection $B \longrightarrow A$, thus if and only if the extension is trivial.

QED

Notice that this cohomological framework does not take into account the full Poisson algebra structure, but simply the Lie algebra one: indeed the cohomology we defined is the Lie algebra cohomology of A with coefficients in E; to let the associative structure play a role as well we have to consider a slightly different cohomology, which we'll introduce again by means of a complex rather than in terms of homological algebra.

Remember that the Poisson structure on A induces a Lie algebra structure on Ω_A:

Definition 6.2 *A representation of a Poisson algebra A is an A-module E (w.r.t. associative structure) which is also a representation of the Lie algebra Ω_A such that*

$$[\omega, ae] = a\{\omega, e\} - \mathbf{i}_{X_a}\omega e$$

for each $\omega \in \Omega_A$, $a \in A$ and $e \in E$, where $[\omega, e]$ stands for the Lie action of Ω_A on E.

If E is a representation of a Poisson algebra A, by putting

$$\{a, e\} = [da, e]$$

we get an action of the Lie algebra A on E:

$$\begin{aligned}
\{\{a, b\}, e\} &= [d\{a, b\}, e] = [\{da, db\}, e] = [da, [db, e]] - [db, [da, e]] \\
&= \{a, \{b, e\}\} - \{b, \{a, e\}\}
\end{aligned}$$

such that E becomes a Poisson module:

$$\{a, be\} = [da, be] = b[da, e] - \{b, a\}e = b\{a, e\} + \{a, b\}e$$

Moreover, notice that if the same Poisson structure $\{\,\}$ on E is induced by the same representations $[\,]$ and $[\,]'$ then

$$[da, e] = \{a, e\} = [da, e]'$$

thus these representations do coincide on exact differentials; of course this *does not imply* that they have to coincide on the all Ω_A (which is generated by the exact differentials as an A-module and not as a $\mathbb{K}$-vector space), unless the following condition is fulfilled

Definition 6.3 *A multiplicative representation of a Poisson algebra A is a representation E of A such that*

$$[a\omega, e] = a[\omega, e]$$

for each $a \in A$, $\omega \in \Omega_A$ and $e \in E$.

If E is multiplicative, as a representation, then the induced structure of Poisson module is multiplicative too (in the sense of Poisson modules):

$$\{ab, e\} = [adb, e] + [bda, e] = a[db, e] + b[da, e] = a\{b, e\} + b\{a, e\}$$

and the representation determines a unique structure of module.

So we have a map

$$\mathbf{d} : \{\text{Representations}\} \longrightarrow \{\text{Poisson modules}\}$$

which, in general, is not surjective. It is indeed clear that a Poisson structure module on E induces on the exact differentials a well-defined action

$$[db, e] = \{b, e\}$$

which is however impossible to extend to a representation of Ω_A: a natural definition would be

$$[adb, e] = a\{b, e\}$$

This position does define a representation of the Lie algebra Ω_A: indeed it turns out that

$$\begin{aligned}
[\{adb, cde\}, m] &= [c\{adb, de\} + a\{b, c\}de, m] \\
&= [ac\{db, de\} - c\{e, a\}db + a\{b, c\}de, m] \\
&= ac\{b, \{e, m\}\} + a\{b, c\}\{e, m\} - ac\{e, \{b, m\}\} - \\
&\qquad - c\{e, a\}\{b, m\} \\
&= [adb, [cde, m]] - [cde, [adb, m]]
\end{aligned}$$

Moreover

$$[adb, ce] = a\{b, ce\} = ac\{b, e\} + a\{b, c\}e = c[adb, e] - (X_c adb)e$$

Notice that such a representation would be multiplicative, by definition:

$$[adb, e] = a\{b, e\} = a[db, e]$$

The rub is that in general Ω_A is not A-free: we could have

$$\omega = \sum_i a_i db_i = \sum_j c_j de_j$$

so that

$$\sum_i a_i\{b_i, e\} = [\omega, e] = \sum_j c_j\{e_i, e\}$$

while it is not true in general that the first and third member of this equality are the same. Thus the representation induced by the Poisson structure is not always well-defined; since it is always multiplicative, it can be defined only starting from a multiplicative Poisson structure on E: so of the map

$$\mathbf{d} : \{\text{Representations}\} \longrightarrow \{\text{Poisson modules}\}$$

we can say that

Proposition 6.4
(1) $\mathbf{d}$ *induces, by restriction, an injective map*

$$\Big\{\text{Multiplicative representations}\Big\} \longrightarrow \left\{\begin{matrix}\text{Multiplicative}\\ \text{Poisson modules}\end{matrix}\right\}$$

(2) *If* Ω_A *is a free* A*-module then* $\mathbf{d}$ *is bijective.*

If E is a representation of the Poisson algebra A, then we can define a cohomology $H^\bullet_\pi(A, E)$ and a homology $H_\bullet^\pi(A, E)$ of A with coefficients in E, as the cohomology and the homology of the Lie algebra Ω_A with coefficients in the representation E. Obviously, in the case $E = A$ and w.r.t. the adjoint representation

$$[\omega, a] = \pi^\#\omega(a)$$

we get the Poisson cohomology and homology as defined by Lichnerowicz and Koszul (cf. [7, §5]).

The representations of a Poisson algebra of course do form a category, whose morphisms are the linear operators

$$f : E \longrightarrow F$$

between representation spaces which are both A-linear and morphisms between representations of the Lie algebra Ω_A:

$$f(ae) = af(e) \qquad \text{and} \qquad f[\omega, e] = [\omega, f(e)]$$

With this definition, the map $\mathbf{d}$ previously considered becomes a covariant functor: indeed if $f : E \longrightarrow F$ is a morphism of representations, then it induces a morphism of modules, since

$$f\{a, e\} = f[da, e] = [da, f(e)] = \{a, f(e)\}$$

The functorial properties of these cohomology and homology are the usual ones: if $f : A \longrightarrow B$ is a morphism of Poisson algebras then it induces a morphism $\Omega f : \Omega_A \longrightarrow \Omega_B$ defined as

$$\Omega f(adb) = f(a)df(b)$$

and such that

$$\Omega f\{adb, cde\} = \{\Omega f(adb), \Omega f(cde)\}$$

So, if E is a representation of B then the Poisson algebra morphism $f : A \longrightarrow B$ induces a representation f^*E of A which, as a vector space is the same, endowed with the actions

$$a \cdot e = f(a) \cdot e \qquad \text{and} \qquad [\omega, e] = [\Omega f(\omega), e]$$

This morphism induces in turn algebra morphisms

$$H^\pi_\bullet(A, f^*E) \longrightarrow H^\pi_\bullet(B, E) \qquad \text{and} \qquad H_\pi^\bullet(B, E) \longrightarrow H_\pi^\bullet(A, f^*E)$$

However, in the geometrical case, the one we are actually interested in, this functoriality is not the “right” one: if $A = C^\infty(M)$ and $B = C^\infty(N)$ are the Poisson algebras of two Poisson manifolds M and N, a Poisson map $F : M \longrightarrow N$ *does not define a morphism of Poisson representations*: indeed A is a representation of $\Omega^1(M)$ and B a representation of $\Omega^1(N)$, but it is not true that $F^*B = A$; this explains why Poisson cohomology, as usually is defined, is not functorial.

Now, we guess, the reader needs some example.

Example 6.5 Der A *is a Poisson representation w.r.t.*

$$[\omega, X] = [\pi^{\#}\omega, X]$$

In fact

$$\begin{aligned}
[\{\omega_1, \omega_2\}, X] &= [\pi^{\#}\{\omega_1, \omega_2\}, X] = [[\pi^{\#}\omega_1, \pi^{\#}\omega_2], X] \\
&= [\pi^{\#}\omega_1, [\pi^{\#}\omega_2, X]] - [\pi^{\#}\omega_2, [\pi^{\#}\omega_1, X]] \\
&= [\omega_1, [\omega_2, X]] - [\omega_2, [\omega_1, X]]
\end{aligned}$$

Leibniz identity is obvious

$$[\omega, aX] = [\pi^{\#}\omega, aX] = a[\pi^{\#}\omega, X] + \pi(\omega \wedge da)X = a[\omega, X] - \mathbf{i}_{X_a}\omega X$$

Notice that this is not a multiplicative representation.

Example 6.6 Ω_A *is a Poisson representation by means of*

$$a \cdot \omega = a\omega \qquad \text{and} \qquad [\omega_1, \omega_2]' = \{\omega_1, \omega_2\}$$

Indeed $[\,]'$ *are Lie brackets and*

$$[\omega_1, a\omega_2]' = \{\omega_1, a\omega_2\} = a\{\omega_1, \omega_2\} + \pi(\omega_1 \wedge da)\omega_2 = a\{\omega_1, \omega_2\} - X_a\omega_1\omega_2$$

Again, this is not a multiplicative representation.

Example 6.7 *Consider another Poisson representation on* Ω_A*:*

$$[\omega_1, \omega_2]'' = \mathcal{L}_{\pi^{\#}\omega_1}\omega_2$$

Since $\mathcal{L}_{[\pi^{\#}\omega_1, \pi^{\#}\omega_2]} = [\mathcal{L}_{\pi^{\#}\omega_1}, \mathcal{L}_{\pi^{\#}\omega_2}]$ *this defines a Lie action, which is Poisson because*

$$[\omega_1, a\omega_2]'' = \mathcal{L}_{\pi^{\#}\omega_1}a\omega_2 = \pi(\omega_1 \wedge da)\omega_2 + a[\omega_1, \omega_2]'' = a[\omega_1, \omega_2]'' - X_a\omega_1\omega_2$$

but, as before, this is not a multiplicative representation.

All these examples satisfy the following

Definition 6.8 *A representation* E *of a Poisson algebra is said to be regular if, for each* $c \in \operatorname{Cas} A$ *and for all* $e \in E$*:* $[dc, e] = 0$.

Of course Der A is regular, because $\pi^{\#}dc = X_c = 0$, and Ω_A is regular (both w.r.t. $[]'$ and to $[]''$) because

$$\{dc, \omega\} = \mathcal{L}_{X_c}\omega - d\pi^{\#}\omega(c) - d\pi(dc \wedge \omega) = -d\pi(\omega \wedge dc) + d\pi(\omega \wedge dc) = 0$$

Obviously these examples correspond to the already known Poisson module structures on Der A and on Ω_A:

$$[da, X] = [X_a, X] = \{a, X\}$$

while both $[]'$ and $[]''$ give rise to the same Poisson structure:

$$[da, \omega]' = \{da, \omega\} = \mathcal{L}_{X_a}\omega - \mathcal{L}_{\pi^{\#}\omega}da - d\pi(da \wedge \omega) = \{a, \omega\} = [da, \omega]'' = \mathcal{L}_{X_a}\omega$$

However non every Poisson module we met till now is induced by some representation: for example the dual (vector space) A' is not a representation w.r.t. the coadjoint actions:

$$[\omega, \varphi] = \varphi \circ \pi^{\#}\omega$$

This is indeed only a skew-representation of the Lie algebra Ω_A

$$\begin{aligned}
[\{\omega_1, \omega_2\}, \varphi] &= \varphi(\pi^{\#}\{\omega_1, \omega_2\})) = \varphi([\pi^{\#}\omega_1, \pi^{\#}\omega_2]) \\
&= \varphi(\pi^{\#}\omega_1(\pi^{\#}\omega_2)) - \varphi(\pi^{\#}\omega_2(\pi^{\#}\omega_1)) \\
&= [\omega_1, \varphi](\pi^{\#}\omega_2) - [\omega_2, \varphi](\pi^{\#}\omega_1) \\
&= [\omega_2, [\omega_1, \varphi]] - [\omega_1, [\omega_2, \varphi]]
\end{aligned}$$

and, above all, it does not satisfy Leibniz identity, since

$$[\omega, a\varphi] = a[\omega, \varphi]$$

7 Connections and Poisson modules

Cohomologies considered so far are essentially two: de Rham cohomology, thus the cohomology of the Lie algebra Der A, and Poisson cohomology, thus the cohomology of the Lie algebra Ω_A; however another cohomology naturally arise in our context: the cohomology with coefficients in the module $\mathcal{H}_A$; this module in some respects resembles Der A, being in fact a submodule of it, in other respects resembles Ω_A, being for instance a differential module.

Suppose E to be a representation of A: we can define

$$[X, e] = [\omega, e]$$

where $\pi^{\#}\omega = X$, using the action of Ω_A on E.

If E is a regular representation then this definition makes sense: indeed from $\pi^{\#}\omega_1 = \pi^{\#}\omega_2 = X$ it follows that $\omega_2 = \omega_1 + \varphi$, where $\varphi \in \ker \pi^{\#}$, hence

$$[\omega_2, e] = [\omega_1, e] + [\varphi, e] = [\omega_1, e]$$

since, for a regular representation, $[dc, e] = 0$ as $c \in \mathrm{Cas\,A}$, and this subalgebra generates the module $\ker \pi^{\#}$.

Therefore a regular representation induces a Lie action of $\mathcal{H}_A$ on E: obviously it is a Lie action

$$\begin{aligned}
[[X_1, X_2], e] &= [[\pi^{\#}\omega_1, \pi^{\#}\omega_2], e] = [\pi^{\#}\{\omega_1, \omega_2\}, e] = [\{\omega_1, \omega_2\}, e] \\
&= [\omega_1, [\omega_2, e]] - [\omega_2, [\omega_1, e]] = [X_1, [X_2, e]] - [X_2, [X_1, e]]
\end{aligned}$$

Leibniz identity for the representation becomes

$$\begin{aligned}
[X, ae] &= [\pi^{\#}\omega, ae] = [\omega, ae] = a[\omega, e] - \mathbf{i}_{X_a}\omega e = a[X, e] + \pi(\omega \wedge a)e \\
&= a[X, e] + \mathbf{i}_{\pi^{\#}\omega} dae = a[X, e] + (Xa)e
\end{aligned}$$

Proposition 7.1 *If E is a representation of the Lie algebra $\mathcal{H}_A$ satisfying Leibniz identity*

$$[X, ae] = a[X, e] + (Xa)e$$

which is also an A-module, then E is induced from a regular representation of A.

PROOF: The only natural thing to do is to define, for $\omega \in \Omega_A$ and $e \in E$

$$[\omega, e] = [\pi^{\#}\omega, e]$$

in this way we get, by the computations just made, a representation of the Lie algebra Ω_A, which satisfies Leibniz identity. Its regularity follows from

$$[dc, e] = [\pi^{\#}dc, e] = [X_c, e] = 0$$

QED

Thus regular representations may be thought as objects defined on $\mathcal{H}_A$.

Example 7.2 *The representation* $\mathrm{Der}\, A$*, which is regular, gives rise to the adjoint representation of the Lie algebra $\mathcal{H}_A$, thus the Lie action is exactly the commutator of a derivation in $\mathcal{H}_A$ with an arbitrary one.*

When Ω_A and the representation is defined as $[\omega_1,\omega_2]=\{\omega_1,\omega_2\}$ we get

$$\begin{aligned}[X,\omega] &= \mathcal{L}_X\omega-\mathcal{L}_{\pi^\#\omega}\pi^{\#-1}X-d\mathbf{i}_{\pi^\#\pi^{\#-1}X}\omega\\ &= \mathcal{L}_X\omega-\mathcal{L}_{\pi^\#\omega}\pi^{\#-1}X-d\mathbf{i}_X\omega\\ &= \mathbf{i}_Xd\omega-d\mathbf{i}_{\pi^\#\omega}\pi^{\#-1}X-\mathbf{i}_{\pi^\#\omega}d\pi^{\#-1}X\end{aligned}$$

while, when the representation is defined as $[\omega_1,\omega_2]=\mathcal{L}_{\pi^\#\omega_1}\omega_2$ we find

$$[X,\omega]=\mathcal{L}_X\omega$$

Next we come to cohomology: if E is a regular representation of A we can consider the complex $C^n(\mathcal{H}_A,E)=\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}_A^n,E)$ equipped with the coboundary maps

$$\begin{aligned}dP(X_0,...,X_k) &= \sum_{i=0}^{k}(-1)^i[X_i,P(X_0,...,\widehat{X_i},...,X_k)]+\\ &\quad+\sum_{i<j}^{0...k}(-1)^{i+j}P([X_i,X_j],X_0,...,\widehat{X_i},...,\widehat{X_j},...,X_k)\end{aligned}$$

The cohomology $H(\mathcal{H}_A,E)$ of this complex is connected to Poisson brackets on A; of course the map $\Omega_A\longrightarrow\mathcal{H}_A$ induces a map in cohomology

$$\pi^*:H(\mathcal{H}_A,E)\longrightarrow H_\nabla(A,E)$$

If the Poisson structure is non-degenerate, then π is an isomorphism, and, *a fortiori*, also π^* is; if the Poisson structure is null then $H_\pi(A,E)$ coincides with the cochain space, while $H(\mathcal{H}_A,E)=0$ (for positive degrees); thus this cohomology is somewhat reduced if compared with Poisson cohomology, hence more simple to compute.

Of course for each differential A-module (D,δ) which is also a Lie algebra and for each Poisson A-module E we can perform a similar construction: in particular we can consider the case $D=\mathcal{H}_A$; remember that in this case the differential map is $X:A\longrightarrow\mathrm{Ham}\,(A)$ extended as

$$X(aX(b))=X(a)\wedge X(b)$$

and that we have a contraction $\mathbf{i}:\mathrm{Ham}\,A\times\mathcal{H}_A\longrightarrow A$ which allows us to define a coboundary map

$$\begin{aligned}\mathbf{i}_{X_0\wedge...\wedge X_n}X(P) &= \sum_{i=0}^{n}(-1)^i\mathbf{i}_{X_i}\mathbf{i}_{X_0\wedge...\widehat{X_i}...\wedge X_n}P\\ &\quad+\sum_{i<j}(-1)^{i+j}\mathbf{i}_{[X_i,X_j]\wedge X_0\wedge...\widehat{X_i}...\widehat{X_j}...\wedge X_n}P\end{aligned}$$

for $P \in \bigwedge^n \operatorname{Ham} A$ and $X_i = X_{a_i} \in \operatorname{Ham} A$.

For example

$$\mathbf{i}_{X_a \wedge X_b} X(P) = \mathbf{i}_{X_a}\mathbf{i}_{X_b} P - \mathbf{i}_{X_b}\mathbf{i}_{X_a} P - \mathbf{i}_{X_{\{a,b\}}} P$$

(remember that $\mathbf{i}_{X_a} X_b = \{a, b\}$).

Of course these cohomologies are connected by a commutative diagram

$$\begin{array}{ccc} H_{dR}(\Omega_A) & \longrightarrow & H_\pi(A) \\ \downarrow & & \downarrow \\ H_{dR}(\mathcal{H}_A) & \longrightarrow & H_\pi(\mathcal{H}_A, A) \end{array}$$

where vertical arrows are induced by $\pi^\# : \Omega_A \longrightarrow \mathcal{H}_A$.

Now, if E is an A-module and $\nabla : E \longrightarrow E \otimes \mathcal{H}_A$ a flat $\mathcal{H}_A$-connection, the spaces $E \otimes \bigwedge \mathcal{H}_A$ defines a complex whose cohomology we denote by $H_\nabla(\mathcal{H}_A, E)$. Yet there's another commutative diagram

$$\begin{array}{ccc} H_\nabla(\Omega_A, E) & \longrightarrow & H_\pi(E) \\ \downarrow & & \downarrow \\ H_\nabla(\mathcal{H}_A, E) & \longrightarrow & H_\pi(\mathcal{H}_A, E) \end{array}$$

Now we consider $\mathcal{H}_A$-connections on A-modules, where A is a Poisson algebra.

Definition 7.3 *A $\mathcal{H}_A$-connection in an A-module E is said to be a Hamiltonian connection.*

Such a connection determines (and is determined by) a *covariant Hamiltonian derivative* $\mathbf{D} : \mathcal{H}_A \longrightarrow \operatorname{End}_{\mathbb{K}}(E)$ such that

$$\mathbf{D}_X(ae) = a\mathbf{D}_X e + X(a)e$$

Suppose the A-module E equipped with such a connection: then if we put

$$\{a, e\} := \mathbf{D}_{X_a} e$$

we get a $\mathbb{K}$-bilinear map $\{\,\} : A \times E \longrightarrow E$ such that

$$\{a, be\} = \mathbf{D}_{X_a}(be) = b\mathbf{D}_{X_a} e + X_a(b)e = b\{a, e\} + \{a, b\}e$$

Furthermore

$$\{ab, e\} = \mathbf{D}_{X_{ab}} e = \mathbf{D}_{aX_b} e + \mathbf{D}_{bX_a} e = a\mathbf{D}_{X_b} e + b\mathbf{D}_{X_a} e = a\{b, e\} + b\{a, e\}$$

Hence, for the brackets $\{\,\}$ to endow E with a multiplicative Poisson module structure it suffices that they define a Lie action: the obstruction to this fact is represented by the vanishing of

$$\begin{aligned} R(a,b)(e) &= \{a,\{b,e\}\} - \{b,\{a,e\}\} - \{\{a,b\},e\} \\ &= \mathbf{D}_{X_a}\mathbf{D}_{X_b}e - \mathbf{D}_{X_b}\mathbf{D}_{X_a}e - \mathbf{D}_{X_{\{a,b\}}}e \\ &= [\mathbf{D}_{X_a},\mathbf{D}_{X_b}] - \mathbf{D}_{[X_a,X_b]} = R_{\mathbf{D}}(X_a,X_b) \end{aligned}$$

and this means that the brackets $\{\,\}$ defines a Lie representation if and only if the connection ∇ is flat.

Now, if two Hamiltonian connections ∇ and ∇' determines the same Poisson structure $\{\,\}$ on E then the A-linear map $\nabla - \nabla' \in \mathrm{End}\,(E)$ is such that, for each $a \in A$:

$$\mathbf{i}_{X_a}(\nabla e - \nabla' e) = 0$$

and so, since the contraction $\mathbf{i} : \mathcal{H}_A \times \mathcal{H}_A \longrightarrow A$ is non-degenerate, $\nabla = \nabla'$. So there exists an injective map

$$\{\text{flat } \mathcal{H}_A\text{-connections}\} \longrightarrow \{\text{multiplicative structures on } E\}$$

In general this map is not surjective, and this amounts to say that not every multiplicative Poisson structure is induced by a flat connection: it suffices to consider non projective modules (so modules without connections): for example on a compact manifold consider the module of distributions $\mathcal{D}(M)'$ (thus of the continuous linear functionals on the Fréchet space $C^\infty(M)$) w.r.t. the coadjoint Poisson structure: if $f,\varphi \in C^\infty(M)$ and $T \in \mathcal{D}(M)'$

$$\{f,T\}(\varphi) = T\{f,\varphi\}$$

This is a multiplicative Poisson structure, but the module $\mathcal{D}(M)'$ is not projective.

To be able to induce a connection from a multiplicative Poisson structure it suffices for example that the module $\mathcal{H}_A$ generated by Hamiltonian derivations is free on A: indeed if $\mathcal{H}_A = A^n$ then we can write every one of its elements X as $X = \sum_i a_i X_{h_i}$ where $a_i, h_i \in A$ are uniquely determined. Then, if $\{\,\}$ is a multiplicative Poisson structure on the module E, we can define a covariant Hamiltonian derivation $\mathbf{D}$ as

$$\mathbf{D}_{\sum_i a_i X_{h_i}} e = \sum_i a_i \{h_i, e\}$$

This makes sense exactly since $\mathcal{H}_A$ is a free module, and defines a $\mathbb{K}$-bilinear map; moreover, if $X \in \mathcal{H}_A$

$$\mathbf{D}_{aX}e = \mathbf{D}_{a\sum_i a_i X_{h_i}}e = \mathbf{D}_{\sum_i aa_i X_{h_i}}e = \sum_i aa_i\{h_i, e\} = a\sum_i a_i\{h_i, e\}$$

Leibniz identity for the covariant derivative comes from

$$\begin{aligned}\mathbf{D}_X ae &= \mathbf{D}_{\sum_i a_i X_{h_i}}ae = \sum_i a_i\{h_i, ae\} = \sum_i a_i(\{h_i, a\}e + a\{h_i, e\}) \\ &= a\sum_i a_i\{h_i, e\} + \sum_i a_i X_{h_i}(a)e = a\mathbf{D}_X e + X(a)e\end{aligned}$$

Finally let's show that the induced connection is flat:

$$\begin{aligned}\mathbf{D}_{[aX_h, bX_k]} &= \mathbf{D}_{ab[X_h, X_k]} + \mathbf{D}_{aX_h(b)X_k} - \mathbf{D}_{bX_k(a)X_h} \\ &= ab\mathbf{D}_{[X_h, X_k]} + aX_h(b)\mathbf{D}_{X_k} - bX_k(a)\mathbf{D}_{X_h} \\ &= ab\mathbf{D}_{X_h}\mathbf{D}_{X_k} + aX_h(b)\mathbf{D}_{X_k} - ab\mathbf{D}_{X_k}\mathbf{D}_{X_h} - bX_k(a)\mathbf{D}_{X_h} \\ &= \mathbf{D}_{aX_h}\mathbf{D}_{bX_k} - \mathbf{D}_{bX_k}\mathbf{D}_{aX_h}\end{aligned}$$

By linearity, the result holds in general; we used the identity $[\mathbf{D}_{X_h}, \mathbf{D}_{X_k}] = \mathbf{D}_{[X_h, X_k]}$ which amounts to claim that the action of the Poisson module is a Lie action:

$$\mathbf{D}_{X_{\{h,k\}}}e = \{\{h, k\}, e\} = \{h, \{k, e\}\} - \{k, \{h, e\}\} = \mathbf{D}_{X_h}\mathbf{D}_{X_k}e - \mathbf{D}_{X_k}\mathbf{D}_{X_h}e$$

Therefore

Theorem 7.4 *If the module $\mathcal{H}_A$ is free on A then there exists a 1-1 correspondence between flat $\mathcal{H}_A$-connections and multiplicative Poisson structures on an A-module.*

The map

$$\{\text{flat } \mathcal{H}_A\text{-connections}\} \longrightarrow \{\text{multiplicative Poisson structures on } E\}$$

splits as

$$\{\text{flat } \mathcal{H}_A\text{-connections}\} \longrightarrow \left\{\begin{array}{c}\text{multiplicative} \\ \text{representations}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{multiplicative} \\ \text{Poisson modules}\end{array}\right\}$$

Indeed a $\mathcal{H}_A$-connection $\nabla : E \longrightarrow E \otimes \mathcal{H}_A$ (or better its covariant derivative) determines a representation E as

$$[\omega, e] = \mathbf{D}_{\pi^\# \omega} e$$

since ($\pi^\#$ is skew-symmetric)

$$[\omega, ae] = \mathbf{D}_{\pi^\# \omega} ae = a[\omega, e] + \mathbf{i}_{\pi^\# \omega} da \otimes e = a[\omega, e] - \mathbf{i}_{X_a} \omega \otimes e$$

obviously it is a multiplicative representation

$$[a\omega, e] = \mathbf{D}_{a\pi^\# \omega} e = a\mathbf{D}_{\pi^\# \omega} e = a[\omega, e]$$

Notice that the map which sends a $\mathcal{H}_A$-connection into a representation is injective but, in general, not surjective: however we can characterize its image as the space of regular multiplicative representations, thus those such that $[\omega, e] = 0$ if $\omega \in \ker \pi^\#$. In fact if $[\,]$ is such a representation then the definition

$$\mathbf{D}_X e = [\pi^{\#-1} X, e]$$

makes sense, since if $\omega \in \pi^{\#-1} X$ then $\omega = \pi^{\#-1} X + \gamma$ where γ is a form which vanishes on an element of the space $\mathcal{H}_A$: it follows, by regularity of the representation, that

$$\mathbf{D}_X e = [\pi^{\#-1} X, e] = [\pi^{\#-1} X + \gamma, e] = \mathbf{D}_{\pi^\# \omega} e$$

Bibliography

[1] R. Abraham, J. Marsden, *Foundations of Mechanics,* Addison–Wesley, New York, 1985.

[2] K.H. Bhaskara, K. Viswanath, *Poisson algebras and Poisson manifolds,* Pitman Res. Notes in Math. **174**, Longman, New-York, 1988.

[3] J. Cuntz, D. Quillen, *Algebra extensions and nonsingularity,* J. Amer. Math. Soc. **8** (1995) 251–289.

[4] A. Cannas da Silva, A. Weinstein, *Geometric Models for Noncommutative Algebras,* American Mathematical Society, Providence, 1999.

[5] I.Ya. Dorfman, I.M. Gel'fand, *Hamiltonian Operators and Algebraic Structures related to them,* Funct. Anal. Appl. **13** (1979), 248–262.

[6] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry,* Springer, New York–Berlin, 1994.

[7] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds,* Progr. Math. **118**, Birkhäuser, Basel, 1994.