Examples of Poisson Modules, II

Paolo Caressa

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Abstract. We give a geometric description of different classes of Poisson modules as introduced in [1]: we start with tensors tangent to leaves on a Poisson manifold, consider Poisson structures on bundles and also an example of Poisson module on a manifold which does not come from any vector bundle; finally we use this language to sketch some integral calculus on Poisson manifolds: we suggest how to introduce integration, homology and cohomology in our setting.

1 Introduction

This note is a geometric continuation, or maybe a geometric version, of [1]: we will adopt the notations and assume the concept of Poisson modules developed therein. In particular we are interested in providing geometric examples of Poisson modules over Poisson algebras: remind that such an algebra is both an associative algebra $(A, \cdot)$ and a Lie algebra $(A, \{,\})$ such that, for each $a, b, c \in A$, the following Leibniz identity holds:

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

Our main definition is the following [1]:

Definition. *A Poisson module over A is an A-module E endowed with a $\mathbb{K}$-linear map $\lambda : A \times E \longrightarrow E$ such that*

$$\lambda(\{a, b\}, e) = \lambda(a, \lambda(b, e)) - \lambda(b, \lambda(a, e))$$

$$\{a, b\} \cdot e = a \cdot \lambda(b, e) - \lambda(b, a \cdot e)$$

for each $a, b \in A$ and $e \in E$ (and $\cdot$ denotes the associative module action).

If moreover the following identity holds

$$\{ab, e\} = a\{b, e\} + b\{a, e\} \tag{M}$$

the module is called *multiplicative*. We deal only with commutative Poisson algebras, namely algebras of smooth functions on manifolds (as well known, a Poisson manifold is precisely a manifold whose algebra of smooth functions is a Poisson algebra), and we use the algebraic machinery set up in [1] to give a geometric description of multiplicative Poisson modules.

In section one we describe the Poisson module of vector fields tangent to the leaves of the generalised foliation induced by a Poisson structure on a manifold. In section two we use the algebraic framework of [1] to relate the structure of multiplicative Poisson module on the sections of a vector bundle to connections on the bundle itself. In section three we give an example of Poisson module which is not the space of sections of any vector bundle, the module of distributions on a Poisson manifolds, and in section four we try to extend this example giving a sketch of integration theory and a cohomology slightly more functorial than the usual Poisson–Lichnerowicz cohomology (cf. [6]) considered on Poisson manifolds.

Notations. We will denote by $C^\infty(M)$ the algebra of real smooth functions on a manifold M, by $\Omega^n(M)$ the space of differential n-forms on M, by $\mathfrak{X}(M)$ the space of vector fields on M; we will denote by $\mathbf{i}_X\varphi$ the contraction of a form φ on a vector field X; moreover we will also write $\langle X, \varphi\rangle$ to mean the pairing between contravariant and covariant tensor fields; $\mathcal{L}_X\varphi$ will denote the Lie derivative of a form along a field.

If M is a Poisson manifold with Poisson brackets $\{,\}$ we denote by $\mathrm{Ham}\,(M)$ the Lie algebra of Hamiltonian vector fields (thus fields of the form $Xf = \{g, f\}$ for some $g \in C^\infty(M)$), by $\mathrm{Can}\,(M)$ the Lie algebra of canonical vector fields (thus fields X such that $X\{f, g\} = \{Xf, g\} + \{f, Xg\}$), by $\mathrm{Cas}\,(M)$ the space of Casimir functions (thus functions $f \in C^\infty(M)$ such that, for each $g \in C^\infty(M)$: $\{f, g\} = 0$); if π is the Poisson tensor, then with $\pi^\#$ we will denote the induced map $\pi^\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$.

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2 Symplectic tensors on a Poisson manifold

If $M = \bigcup_x S_x$ is the decomposition in symplectic leaves of a Poisson manifold M, then we can consider the immersions $i : S_x \hookrightarrow M$ and try to transfer the usual concepts given on the symplectic manifold S_x to the Poisson manifold M: it is not immediately clear how this can be done; for example neither a differential form nor a vector field on S_x do project on M. So we are forced to give the following

Definition 2.1 *A symplectic vector field on a Poisson manifold M is a vector field $X \in \mathfrak{X}(M)$ such that for each $x \in M$, if S_x is the symplectic leaf containing x, then $X|_{S_x} \in \mathfrak{X}(TS_x)$. We denote the space of symplectic fields with as $\mathfrak{S}(M)$.*

In other words, symplectic vector fields are precisely vector fields tangent to leaves: of course on a symplectic manifold we have $\mathfrak{S}(S) = \mathfrak{X}(S)$, while on a Poisson manifold equipped with the null Poisson tensor we have $\mathfrak{S}(M) = 0$, since every point is a single symplectic leaf. Notice that if $i_S : S \longrightarrow M$ is the injection of a leaf into M then, for $s \in S$:

$$(X|_S)_s = X_{i(s)}$$

hence, if X is symplectic, $Xf(s) = \langle (df)_S, X_s \rangle$.

We are interested in symplectic fields first of all because they include Hamiltonian vector fields: more precisely, since symplectic vector fields, as a $C^\infty(M)$-module, are generated by the Lie algebra of Hamiltonian fields, as follows by definition, we have

Proposition 2.2 *The space $\mathfrak{S}(M)$ of symplectic vector fields is precisely the Poisson module generated by Hamiltonian fields.*

However, a canonical field is not necessarily symplectic: it suffices to consider a Poisson manifold with the zero Poisson tensor to get a manifold where each field is canonical, but only the null field is symplectic; moreover a symplectic field is not necessarily locally Hamiltonian, as showed by the following example.

Example 2.3 *On the Poisson plane $\mathbb{R}^2_\pi$ (on the plane a Poisson structure is determined by a smooth function π) symplectic fields must vanish at point of null rank. Consider in particular the Poisson plane $\mathbb{R}^2_\pi$ with $\pi(x,y) = y^2$: its symplectic leaves are the upper and lower half-planes, and each single point of the line $\{y = 0\}$; the field $y\partial_x$ is obviously symplectic: indeed its restriction to the upper (or to the lower) half-plane defines a vector field in that half-plane, and the field vanishes on each point of the line $\{y = 0\}$; however, even if it is symplectic, this field is not locally Hamiltonian, since*

$$y\partial_x = X_f \quad \Longrightarrow \quad y\frac{\partial g}{\partial x} = y^2\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)$$

(for every g), which implies that $\frac{\partial f}{\partial x} = 0$ and $y\frac{\partial f}{\partial y} = -1$, therefore, around a singular point of the line $\{y = 0\}$, the field can't be Hamiltonian.

The space of symplectic fields will play the role of the space of differential 1-forms in the symplectic case: for example, if S is a symplectic manifold, then $\mathfrak{S}(S) = \mathfrak{X}(S)$, and so, by means of the isomorphism induced by the symplectic form, $\mathfrak{S}(S) \cong \Omega^1(S)$ as $C^\infty(M)$-modules. Recall that the Lie action of $C^\infty(M)$ on the Poisson module $\mathfrak{S}(M)$ is given by

$$\{f, X\} = [X_f, X]$$

In the symplectic case we remark that, by applying the isomorphism $\pi^\#$:

$$\{f, \pi^\#\omega\} = [X_f, \pi^\#\omega] = \pi^\#\{df, \omega\}$$

Hence the Poisson module $\mathfrak{S}(M)$ is isomorphic to $\Omega^1(M)$ w.r.t. the usual Poisson structure.

Moreover notice that if M and N are Poisson manifolds and $X \in \mathfrak{S}(M \times N)$ then $X|_S \in \mathfrak{X}(S)$ for each symplectic leaf in $M \times N$ (w.r.t. the usual Poisson product structure, cf. [6, §1]); but $S = S_N \times S_M$ and so $\mathfrak{X}(S) \cong \mathfrak{X}(S_N) \oplus \mathfrak{X}(S_M)$; it follows

Proposition 2.4 $\mathfrak{S}(M \times N) \cong \mathfrak{S}(N) \oplus \mathfrak{S}(N)$

The definition of a symplectic vector field extends in an obvious way to any tensor field: tensors we are interested in are polyvector fields (contravariant skew-symmetric tensor fields).

Definition 2.5 *A symplectic p-tensor is a polyvector field of degree p which, whenever restricted to any symplectic leaf, belongs to the space of tensor fields of that leaf. The space of symplectic tensors of degree p will be denoted by $\mathfrak{S}^p(M)$.*

Of course

Lemma 2.6 *The Poisson tensor π is a symplectic 2-tensor.*

Now we want to present geometrically the space $\mathfrak{S}(M)$, as the image of the map $\pi^\# : T^*M \longrightarrow TM$ induced by the Poisson structure: to do it simply notice that

Lemma 2.7 *$X \in \mathfrak{S}(M)$ if and only if, for each differential form $\alpha \in \Omega^1(M)$ such that in each symplectic leaf $S \subset M$ from $s^*\alpha = 0$ it follows $\mathbf{i}_X\alpha = 0$, where $s : S \longrightarrow M$ is the injection of S into M and $s^*\alpha$ is the pull-back of differential forms.*

Proof: If $X \in \mathfrak{S}(M)$ and α is such that $s^*\alpha = 0$ for every S, then, if $x \in M$ and S_x is the leaf containing x:

$$\mathbf{i}_X\alpha(x) = \alpha_x(X_x) = \alpha_x((X|_S)_x) = 0$$

(since $(X|_S)_x \in T_xS_x$). Vice versa suppose that X satisfies to the condition of the lemma and that $x \in S \subset M$; then, if $X|_S(x)$ does not belong to T_xS_x, it would exist a differential form $\alpha \in \Omega^1(M)$ such that $\alpha_x \in T_x^*S_x$ would not vanish when computed on the vector $(X|_{S_x})_x$ but at the same time such that $s^*\alpha = 0$, whence $\mathbf{i}_X\alpha(x) \neq 0$, which is absurd.

□

Theorem 2.8 $\operatorname{Im}\pi^\# = \mathfrak{S}(M)$.

Proof: If $X \in \operatorname{Im}\pi^\#$ then $\mathbf{i}_\alpha X = \pi(\beta_X, \alpha)$ for each $\alpha \in \Omega^1(M)$ and for some $\beta_X \in \Omega^1(M)$, thus, by lemma 2.0., $X \in \mathfrak{S}(M)$.

Vice versa, if $i_S : S \longrightarrow M$ is a symplectic leaf and $\sigma = \pi|_S$ its Poisson (invertible) tensor, the following diagram (which makes sense because $\operatorname{Im}\pi^\# \subset \mathfrak{S}(M)$) commutes:

$$\begin{array}{ccc} \Omega^1(M) & \xrightarrow{\pi^\#} & \mathfrak{S}(M) \\ \downarrow & & \downarrow \\ \Omega^1(S) & \xrightarrow{\sigma^\#} & \mathfrak{X}(S) \end{array}$$

(vertical arrows are the pull-back of 1-forms and the restriction of vector fields). Therefore to each field $X \in \mathfrak{S}(M)$ we can associate a 1-form $\beta_X = \sigma^\flat(X|_S)$ on S and, since the manifold is the union of its symplectic leaves, we can lift anyone of these forms to the form $\alpha \in \Omega^1(M)$ (which is smooth since the foliation is) which induces, by pull-back, every such β_X. Now we use the commutativity of the previous diagram to infer that, for every leaf S,

$$(\pi^\#\alpha)|_S = \sigma^\#(i_S^*\alpha)$$

thus $\pi^\#\alpha = X$.

□

In some sense, symplectic fields have an ambiguous behaviour: on symplectic leaves they correspond to differential forms, by means of the isomorphism between 1-forms and fields induced by the symplecticity of the leaf, and on the other hand they remains vector fields on the manifold M: for instance, if X is a symplectic vector field on M, we can consider its flow, thus the 1-parameter family of local diffeomorphisms $\{\Phi_t\}$ around each $x \in M$ associated to X. Since, if $x_0 \in M$, in a suitable neighbourhood U_0 of x_0 the differential equation

$$(t \in I) \qquad X(c(t)) = \dot{c}(t)$$

with initial data $X(0) = \dot{c}(0)$ has a unique solution (c, I), we can ask how being X symplectic is reflected on c; of course we can consider $U_0 = S_0 \times N_0$, and so a pair of curves (c_S, c_N) determined by the maximal integral curve c of X on U_0. But, since X is symplectic, $X(x_0)$ belongs to the tangent space in x_0 to the leaf $S_0 \times \{x_0\}$, and this means that the tangent vector in x_0 to the integral curve c is tangent to the leaf. Therefore the tangent vector to the leaf is pointwise symplectic and, since the space of symplectic fields is generated by Hamiltonian ones, the curve is piecewise Hamiltonian, thus splits in curves whose tangent vectors are Hamiltonian.

In other words, integral curves of a symplectic field are precisely curves which make points they pass through belonging to a same leaf. This implies that the curve is completely contained in a single leaf, and thus that the flow Φ_t defined as

$$\Phi_t(x_0) = c(t)$$

is actually a local diffeomorphism of the leaf into itself.

Notice that this flow does not preserve the symplectic structure of the leaf, unless the field X is canonical: indeed, in this case, if ω is the symplectic structure obtained by restriction of the Poisson structure of M on the leaf (thus $\omega^\# = -(\pi|_S)^\#$) we find that

$$\frac{d}{dt}\Phi_t^*\omega = \Phi_t^*\mathcal{L}_X\omega = 0$$

and so $\Phi_t^*\omega = \omega$. This will be, in general, false.

Theorem 2.9 *A vector field X on M is symplectic if and only if its flow is determined by a family of local diffeomorphisms of the symplectic leaves.*

Finally let us give a geometrical characterisation of symplectic tensors on a regular Poisson manifold: take a connection $\nabla : \mathfrak{X}(M) \longrightarrow \Omega^1(M) \otimes \mathfrak{X}(M)$ in the tangent bundle of M. The condition

$$\nabla\pi = 0$$

seems to be the most natural compatibility condition between the Poisson structure and the connection. Locally this condition can be written as follows: fix a Darboux–Weinstein coordinate system[1] $(x^1, ..., x^n)$ and its associated bases of vector fields $(\partial_1, ..., \partial_n)$ and of differential forms $(dx^1, ..., dx^n)$; if

$$\pi = \pi^{ij}\partial_i \wedge \partial_j$$

(of course we adopt Einstein's convention) and

$$\nabla\partial_i = \Gamma^k_{ij}\partial_k \otimes dx^j$$

then

$$\begin{aligned}
\nabla\pi &= \pi^{rs}\nabla(\partial_r \wedge \partial_s) + \partial_i \wedge \partial_j \otimes d\pi^{ij} \\
&= \pi^{rs}\Gamma^i_{rk}\partial_i \wedge \partial_s \otimes dx^k + \pi^{rs}\Gamma^j_{sk}\partial_r \wedge \partial_j \otimes dx^k + \partial_k\pi^{ij}\partial_i \wedge \partial_j \otimes dx^k \\
&= \pi^{rj}\Gamma^i_{rk}\partial_i \wedge \partial_j \otimes dx^k + \pi^{is}\Gamma^j_{sk}\partial_i \wedge \partial_j \otimes dx^k + \partial_k\pi^{ij}\partial_i \wedge \partial_j \otimes dx^k \\
&= \left(\pi^{is}\Gamma^j_{sk} - \pi^{jr}\Gamma^i_{rk} + \partial_k\pi^{ij}\right)\partial_i \wedge \partial_j \otimes dx^k
\end{aligned}$$

Thus the condition $\nabla\pi = 0$ is equivalent to the local equations

$$\partial_k\pi^{ij} = \pi^{jr}\Gamma^i_{rk} - \pi^{is}\Gamma^j_{sk}$$

or, in matrix form:

$$\partial_k\pi = \pi\Gamma_{(k)} - \Gamma^T_{(k)}\pi^T$$

We can apply it to recover a well known result:

Theorem 2.10 *A Poisson manifold (M, π) admits a torsionless connection ∇ such that $\nabla\pi = 0$ if and only if M is regular.*

[1] We are using Weinstein splitting theorem: in the regular case this was already settled by Sophus Lie, cf. [7].

Indeed a Poisson manifold is regular if and only if there are local coordinates (around each point) in which the Poisson tensor is constant; in this case equations $\nabla\pi = 0$ become

$$\pi\Gamma_{(k)} = \Gamma_{(k)}^T \pi^T$$

But these are easily satisfied by taking into account that, in the Darboux–Weinstein coordinates we have chosen

$$(i = 1, ..., r) \qquad \pi^{i,i+r} = 1 \quad \text{and} \quad \pi^{i+r,i} = -1$$

and all the other components do vanish ($2r$ is the rank of the Poisson structure in the local neighbourhood). Hence we can determine Christoffel's symbols Γ_{ij}^k which satisfy the previous differential equations exactly in the regular case: it suffices to consider a symplectic connection in the symplectic half of the neighbourhood and any torsionless connection in the singular part.

Now: if M is a regular Poisson manifold, we can assume that there exists a torsionless connection such that $\nabla\pi = 0$; since the map $\pi^\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ is a bundle morphism, the image $\mathfrak{S}(M)$ is a (regular) distribution whose integral leaves are exactly the symplectic leaves of M.

But ∇ is a connection according to which the Poisson tensor is parallel: so if X is a symplectic field

$$\nabla X = \nabla\pi^\#\omega = 0$$

Vice versa, if X is parallel then its restriction to a leaf defines a field on the leaf, thus X is symplectic.

3 Poisson bundles

Be $E \longrightarrow M$ a vector bundle on M and $\Gamma(E)$ its module of smooth global sections: a structure of Poisson module on $\Gamma(E)$ is given by an action of the Poisson algebra $C^\infty(M)$ on the $\Gamma(E)$ by means of a Lie action

$$\{\,\} : C^\infty(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

such that ($f, g \in C^\infty(M)$ and $s \in \Gamma(E)$)

$$\{f, gs\} = g\{f, s\} + \{f, g\}s$$

If $(U; x_1, ..., x_n)$ is a local chart of M which trivialises the bundle E, we can consider a $C^\infty(U)$-basis of the module of local sections $(e_1, ..., e_n)$, so that

$$\{f, e_i\} = \sum_j G^{ij}(f) e_j$$

where $G^{ij} : C^\infty(U) \longrightarrow C^\infty(U)$. Of course the operators G^{ij} are linear, since $\{f + g, e\} = \{f, e\} + \{g, e\}$; more precisely they are $\operatorname{Cas} C^\infty(U)$-linear.

Proposition 3.1 *The matrix of operators G defines a Poisson module structure if and only if*

$$G^{ij}(\{f,g\}) = [G(g), G(f)]^{ij} + \{G^{ij}(f), g\} + \{f, G^{ij}(g)\}$$

(A^{ij} is the entry at the i-th row and j-th column in the matrix A).

Proof: It suffices to notice that

$$\begin{aligned}\{f,\{g,e_i\}\} &= \{f, G^{ij}(g)e_j\} = G^{ij}(g)\{f,e_j\} + \{f, G^{ij}(g)\}e_j \\ &= G^{ij}(g)G^{jk}(f)e_k + \{f, G^{ij}(g)\}e_j\end{aligned}$$

which implies that

$$\{\{f,g\},e_i\} = \{f,\{g,e_i\}\} - \{g,\{f,e_i\}\}$$

is equivalent to the stated condition.

□

Recall that we call multiplicative a Poisson module in which the following identity holds:

$$\{ab, e\} = a\{b,e\} + b\{a,e\} \tag{M}$$

Proposition 3.2 *The Poisson structure on E is multiplicative if and only if G^{ij} are vector fields.*

Proof: Indeed

$$\sum_j G^{ij}(fg)e_j = \{fg, e_i\} = f\{g,e_i\} + g\{f,e_i\} = \sum_j \left(gG^{ij}(f) + fG^{ij}(g)\right) e_j$$

so that G^{ij} is a derivation of the algebra $C^\infty(U)$, hence a local vector field.

□

Notice that we can write a more particular functions matrix by evaluing the entries G^{ij} on the coordinate functions x_k obtaining in such a way n^3 functions

$$G^{ij}_k = G^{ij}(x_k)$$

which put us in grade to reformulate condition 3.0 as

$$G^{ij}(\pi_{rs}) = [G_s, G_r]^{ij} + \pi_{rk}\frac{\partial G^{ij}_s}{\partial x_k} + \pi_{sk}\frac{\partial G^{ij}_r}{\partial x_k}$$

These are not the components of any tensor: this is easily seen with an example: if $E = TM$ has the Poisson structure

$$\{f, X\} = [X_f, X]$$

we easily get that

$$G^{ij}(f) = -\frac{\partial \pi_{rj}}{\partial x_i}\frac{\partial f}{\partial x_r} - \pi_{rj}\frac{\partial^2 f}{\partial x_i x_r}$$

where π_{ij} are the components of the Poisson tensor in the fixed local coordinates, hence

$$G_k^{ij} = -\frac{\partial \pi_{ij}}{\partial x_k}$$

In particular, if M is regular, we can find local coordinates such that $G_k^{ij} = 0$; in general, the matrix G_k will split in the direct sum of a null matrix and an arbitrary matrix satisfying to the written condition, according to Weinstein's splitting theorem; in the symplectic case, w.r.t. Darboux coordinates $(q_1, ..., q_n, p_1, ..., p_n)$ we find that

$$G^{ij}(f) = (-1)^\sigma \frac{\partial^2 f}{\partial x_i x_\alpha}$$

where $(x_1, ..., x_{2n}) = (q_1, ..., p_n)$, α is the index $1 + n + j$ modulo $n + 1$ and the sign $(-1)^\sigma$ is negative when $i, \alpha < n$.

Notice that also in the example $E = T^*M$ with Poisson structure

$$\{f, \omega\} = \mathcal{L}_{X_f}\omega$$

we have, locally

$$G^{ij}(f) = \frac{\partial \pi_{ki}}{\partial x_j}\frac{\partial f}{\partial x_k} + \pi_{ki}\frac{\partial^2 f}{\partial x_k x_j}$$

since $\mathcal{L}_{X_f} dx_i = d\{f, x_i\}$. Again we get the derivatives of the components of the Poisson tensor

$$G_k^{ij} = \frac{\partial \pi_{ij}}{\partial x_k}$$

The coordinate change affects the matrix of operators G^{ij} in the following way: suppose that $(V; y_1, ..., y_n)$ is another local chart and consider $U \cap V$: here we have our G^{ij} associated to the trivialisation of bundle E with coordinates $(U; x)$; one passes from these coordinates to the trivialisation with coordinates (V, y) by means of the cocycle formula which indeed does characterise vector bundles, and which in matrix notation reads as

$$e^V = C_U^V e^U$$

where $e^U = (e_1, ..., e_n)$ is the basis of the module $\Gamma(U \cap V, E)$ w.r.t. (U, x), e^V is the basis w.r.t. (V, y) and $C_U^V : U \cap V \longrightarrow GL_n(\mathbb{R})$ is the bundle cocycle (we can consider the components C_i^j of this matrix as smooth functions on $U \cap V$). In terms of components:

$$e_i^V = C_i^j e_j^U$$

Then

$$\begin{aligned} G_V^{ij}(f)e_j^V &= \{f, e_i^V\} = \{f, C_i^j e_j^U\} = C_i^j\{f, e_j^U\} + \{f, C_i^j\}e_j^U \\ &= C_i^j G_U^{jk}(f)e_k^U + \{f, C_i^j\}e_j^U \end{aligned}$$

thus

$$G_V^{ij}(f)C_j^k e_k^U = C_i^j G_U^{jk}(f)e_k^U + \{f, C_i^j\}e_j^U$$

In terms of matrices: $(C_U^V = (C_V^U)^{-1})$

Proposition 3.3 $G_V(f) = C_U^V G_U(f) C_V^U + \{f, C_U^V\}C_V^U$

So the matrix G does not transform according to a tensorial law, because of the presence of the second summand (in effects when it is possible to reduce the structure group of the bundle in such a way that the coefficients of the cocycle matrices are Casimir functions, we actually have a tensorial behaviour of the $G(f)$). However, since the obstruction to tensoriality of G does not depends on the cocycle, we get

Corollary 3.4 *The difference between two structure of Poisson module is an endomorphism of the bundle E.*

Notice that, in general, these "structure constants" for the Poisson action will be $\mathbb{R}$-linear operators $G^{ij} : C^\infty(U) \longrightarrow C^\infty(U)$ (fulfilling condition 3.0).

A particular case is given for instance when $G^{ij} \in \mathrm{Can}\,(U)$ (canonical vector fields): then, by proposition 3.0, the only condition on the matrix G to define a Poisson structure is

$$[G(f), G(g)] = 0$$

This structure, as we know, is necessarily multiplicative.

Now consider a Poisson representation on the module E: remind (cf. [1]) that such a representation is a representation of the Lie algebra $\Omega^1(M)$ on $\Gamma(E)$ such that

$$[\omega, ae] = a[\omega, e] - \mathbf{i}_{X_a}\omega e$$

Locally we write

$$[\omega, e_i] = H^{ij}(\omega)e_j$$

where $H^{ij} : \Omega^1(U) \longrightarrow C^\infty(U)$. Therefore the definition of a representation becomes

$$H^{ij}(\{\omega, \varphi\}) = [H(\varphi), H(\omega)]^{ij} + \mathbf{i}_{X_{H^{ij}(\omega)}}\varphi - \mathbf{i}_{X_{H^{ij}(\varphi)}}\omega$$

thus, evaluing on exact forms $\{dx_i\}$ which locally generate the module of differentials:

$$\begin{aligned} H^{ij}(d\pi_{rs}) &= [H_s, H_r]^{ij} + \{H_r^{ij}, x_s\} + \{x_r, H_s^{ij}\} \\ &= [H_s, H_r]^{ij} + \pi_{ks}\frac{\partial H_r^{ij}}{\partial x_k} + \pi_{rk}\frac{\partial H_s^{ij}}{\partial x_k} \end{aligned}$$

with $H_k^{ij} = H^{ij}(dx_k)$.

As shown in [1] a representation induces a module, and by comparing the latter equation with the one given for the G_k^{ij} we find that the relationship between the representation and the module induced by it is simply

$$G^{ij} = H^{ij} \circ d$$

Moreover is obvious that the representation is multiplicative if and only if $H^{ij} : \Omega(U) \longrightarrow C^\infty(U)$ are $C^\infty(U)$-linear, and in this case also G^{ij} (induced by H^{ij}) give rise to a multiplicative module:

$$\begin{aligned} G^{ij}(fg) &= H^{ij}(d(fg)) = H^{ij}(fdg) + H^{ij}(gdf) = fH^{ij}(dg) + gH^{ij}(df) \\ &= fG^{ij}(g) + gG^{ij}(f) \end{aligned}$$

Also for representations we can write a formula for the coordinate change in terms of the bundles' cocycle: again we proceed by considering two local charts (U, x) and (V, y) which trivialise the bundle and whose open sets are not disjoint ($U \cap V \neq \emptyset$), and the associated basis of the module of global sections e^U and e^V in the chosen charts. Then

$$\begin{aligned} H_V^{ij}(\omega)e_j^V &= [\omega, e_i^V] = [\omega, C_i^j e_j^U] = C_i^j[\omega, e_j^U] - \mathbf{i}_{X_{C_i^j}}\omega e_j^U \\ &= C_i^j H_U^{jk}(\omega)e_k - \mathbf{i}_{X_{C_i^j}}\omega e_j^U \end{aligned}$$

so that

$$H_V(\omega) = C_U^V H_U(\omega) C_V^U - \mathbf{i}_{X_{C_U^V}} \omega C_V^U$$

Now suppose our Poisson representation to be multiplicative:

$$[a\omega, e] = a[\omega, e]$$

This, locally, means that the maps $H^{ij} : \Omega^1(U) \longrightarrow C^\infty(U)$ are $C^\infty(U)$-linear, hence they correspond to vector fields X^{ij} in U, therefore the representation is determined (locally) by a matrix of vector fields which satisfy the following equation

$$\mathbf{i}_{X^{ij}}\{\omega, \varphi\} = [X(\varphi), V(\omega)]^{ij} + \mathbf{i}_{\pi^\#\omega} d\mathbf{i}_{X^{ij}}\varphi - \mathbf{i}_{\pi^\#\varphi} d\mathbf{i}_{X^{ij}}\omega$$

In other words, to give a multiplicative Poisson representation is the same as to choose in each coordinate system a matrix of vector fields. So, if we consider the functions

$$X_k^{ij} = X^{ij}(x_k)$$

we can use them to define an operator

$$\nabla e_i = X_k^{ij} dx_k \otimes e_j$$

We deduce the coordinate transformation formula for this operator from that of H^{ij}:

$$X_V = C_V^U X_U C_U^V - (\pi^\# dC_V^U) C_U^V$$

Theorem 3.5 *$X^{ij}(dx_k)$ are Christoffel's symbols for a Hamiltonian connection.*

The idea of the proof is to compare our formulas with the transformation rule which characterises Christoffel's symbols (cf. e.g. [2, Vol. I, §III-7.3]). In effects a Hamiltonian connection is an operator $\nabla : \Gamma(E) \longrightarrow \mathfrak{S}(M) \otimes \Gamma(E)$ such that ($f \in C^\infty(M)$, $e \in \Gamma(E)$)

$$\nabla(fe) = f\nabla e + X_f \otimes e$$

Locally, such a connection can be written as

$$\nabla e_i = \Gamma_{ij}^k X_{x_k} \otimes e_j$$

Now consider two local charts $(U; x)$ and $(V; y)$ in which the bundle E is trivial (and such that $U \cap V \neq \emptyset$): first of all we write the equation which connects two local bases in these two different charts

$$e_i^V = C_i^j e_j^U$$

and apply it to our operators

$$\nabla e_i^V = C_i^j \nabla e_j^U + X_{C_i^j} e_j^U = C_i^j \Gamma_{jk}^l X_{x_l} \otimes e_k^U + X_{C_i^j} \otimes e_j^U$$

Therefore, if $\widetilde{\Gamma}_{ij}^k$ are Christoffel's symbols w.r.t. the chart (V, y), we get

$$\widetilde{\Gamma}_{ij}^k C_j^l X_{x_k} \otimes e_l^U = C_i^h \Gamma_{hl}^k X_{x_k} \otimes e_l^U + X_{C_i^l} \otimes e_l^U$$

In a more compact way:

$$\Gamma^V = C_V^U \Gamma^U C_U^V + (\pi^\# dC_V^U) C_U^V$$

This is the transformation law of Christoffel's symbols for a Hamiltonian connection: it is the same formula verified by the functions X_k^{ij} associated to the multiplicative Poisson representation.

Thus we have characterised those representations induced by Hamiltonian connections

Theorem 3.6 *A Poisson representation is induced by a Hamiltonian connection if and only if it is regular and multiplicative.*

Indeed in this case the functions H^{ij} are determined by H_k^{ij} as

$$H^{ij}(\sum_k a_k dx_k) = \sum_k a_k H_k^{ij}$$

as it follows from the multiplicativity of the representation.

Now consider a connection $\nabla : \Gamma(E) \longrightarrow \Omega^1(M) \otimes \Gamma(E)$ in E: locally, once a basis for the module of sections is fixed, we can write Christoffel's symbols of this connection as

$$\nabla e_i = \Gamma_{ij}^k dx_k \otimes e_j$$

and its curvature is the tensor (cf. [1])

$$\begin{aligned} \nabla^2 e_i &= \nabla \Gamma_{ij}^k dx_k \otimes e_j = \Gamma_{ij}^k dx_k \wedge \nabla e_j - d(\Gamma_{ij}^k dx_k) \otimes e_j \\ &= \Gamma_{ij}^k dx_k \wedge \Gamma_{jr}^s dx_s \otimes e_r - d\Gamma_{ir}^k \wedge dx_k \otimes e_r \end{aligned}$$

If ∇ is such a connection, we can use the isomorphism $\pi^\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ to define a map $\Delta : \Gamma(E) \longrightarrow \mathfrak{X}(M) \otimes \Gamma(E)$ as ($\mathbf{I}$ is the identity $\Gamma(E) \longrightarrow \Gamma(E)$)

$$\Delta = \pi^\# \otimes \mathbf{I} \circ \nabla$$

Notice that, being $\pi^\#$ defined also for exterior powers of these modules (cf. [1]), it makes sense to consider the map $\Delta^2 = \pi^\# \otimes \mathbf{I} \circ \nabla^2$ which, by using the local structure equation just recalled, locally can be written as

$$\begin{aligned}\Delta^2 e_i &= \pi^\#(\Gamma_{ij}^k dx_k \wedge \Gamma_{jr}^s dx_s - d\Gamma_{ir}^k \wedge dx_k) \otimes e_r \\ &= (\Gamma_{ij}^k X_{x_k} \wedge \Gamma_{jr}^s X_{x_s} - X_{\Gamma_{ir}^k} \wedge X_{x_k}) \otimes e_r\end{aligned}$$

Δ is a Hamiltonian connection, since

$$\Delta ae = \pi^\# \otimes \mathbf{I}(a\nabla e) + \pi^\# \otimes \mathbf{I}(da \otimes e) = a\pi^\# \otimes \mathbf{I}\nabla e + X_a \otimes e = a\Delta e + X_a \otimes e$$

Thus we have a canonical way to build Hamiltonian connections (and so representations of Poisson modules) starting with ordinary connections in the bundle E.

Vice versa, consider a Hamiltonian connection $\Delta : \Gamma(E) \longrightarrow \mathfrak{S}(M) \otimes \Gamma(E)$: if $\sigma : \mathfrak{S}(M) \longrightarrow \Omega^1(M)$ is the left inverse to the map $\pi^\#$ (thus $\sigma\pi^\# =$ identity), then

$$\nabla := \sigma \otimes \mathbf{I} \circ \Delta$$

is a connection: indeed

$$\nabla ae = \sigma \otimes \mathbf{I}(a\Delta e) + \sigma \otimes \mathbf{I}(X_a \otimes e) = a\nabla e + \sigma\pi^\# da \otimes e = a\nabla e + da \otimes e$$

Of course if the Hamiltonian connection is of the form $\pi^\# \otimes \mathbf{I}\nabla$ then its associated connection is ∇ itself, hence a unique connection may give rise to different Hamiltonian connections, and the set of such Hamiltonian connections is parametrised by left inverses to the module morphism $\pi^\#$.

Since it is a linear map between modules, σ is completely determined on Hamiltonian vector fields, as

$$\sigma X_f = \sigma\pi^\# df = df$$

Moreover

$$(\dagger) \qquad \sigma[X_f, X_g] = \sigma X_{\{f,g\}} = d\{f,g\} = \{df, dg\}$$

so that

Theorem 3.7 *There exists a bijective map*

$$\{\text{Hamiltonian connections}\} \longleftrightarrow \frac{\{\text{Connections}\}}{\left\{\begin{array}{c}\sigma : \mathfrak{S}(M) \longrightarrow \Omega(M) \text{ left inverse} \\ \text{to } \pi^\# \text{ such that } (\dagger) \text{ holds}\end{array}\right\}}$$

4 The module of distributions

We consider in this section an example of multiplicative Poisson module whose Poisson structure does not come from any connection.

Be M a Poisson manifold and $A = C^\infty(M)$ its Poisson algebra: the latter contains a well known ideal, namely smooth functions with compact support: $C_c^\infty(M)$. This is an ideal w.r.t. the associative structure of the Poisson algebra, as it follows from $\operatorname{supp} fg \subset \operatorname{supp} f \cap \operatorname{supp} g$; but Leibniz's identity which holds for Poisson brackets implies the same also for the Lie structure:

$$\operatorname{supp}\{f,g\} \subset \operatorname{supp} f \cap \operatorname{supp} g$$

Notice that the algebra $C_c^\infty(M)$ is a "sub-object" of $C^\infty(M)$ from the algebraic viewpoint but not from the topological viewpoint, since it is not a closed subspace (in the Fréchet topology) being dense; however it is the correct space in which consider test functions for distributions (cf. [4, §I-2]): let us denote by $\mathcal{D}(M)'$, or simply by $\mathcal{D}'$, the set of distributions on M. As it is well known, it is a module over the associative algebra $C^\infty(M)$ w.r.t. the coadjoint action: therefore it seems natural, on our context, to define the Poisson brackets between a function $f \in C^\infty(M)$ and a distribution $T \in \mathcal{D}(M)'$ to be the distribution defined as

$$\{f,T\}(\varphi) = T\{f,\varphi\}$$

for each $\varphi \in C_c^\infty(M)$.

In this way we still get a distribution, since it is a continuous (by continuity of differential operators w.r.t. Fréchet topology in $C^\infty(M)$, cf. [4, §III-5]) linear (by bilinearity of Poisson brackets) functional; moreover it is a well defined distribution since if $\varphi \in \mathcal{D}(M)$ and $f \in \mathcal{E}(M)$ then $\{f,\varphi\} \in \mathcal{D}(M)$ (Leibniz identity), so that $\{T,f\} \in \mathcal{D}'$.

Proposition 4.1 *If M is a Poisson manifold then $\mathcal{D}(M)'$ is a multiplicative Poisson module w.r.t. the coadjoint action.*

PROOF: The structure of associative and Lie module is given by the actions

$$(fT)(\varphi) = T(f\varphi) \qquad\qquad \{f,T\}(\varphi) = T(\{\varphi,f\})$$

($T \in \mathcal{D}(M)'$, $f \in C^\infty(M)$ and $\varphi \in C_c^\infty(M)$). Notice that these are well defined actions of $C^\infty(M)$ on $\mathcal{D}(M)'$; furthermore, since derivation and multiplication are continuous in Fréchet topology, $\{f,T\}$ is actually an element of $\mathcal{D}(M)'$ if $f \in C^\infty(M)$ and $T \in \mathcal{D}(M)'$.

That $\mathcal{D}(M)'$ is a module over the associative algebra $C^\infty(M)$ is well known (cf. [4, §V]); let us show that it is a Poisson module: first of all

$$\begin{aligned}\{\{f,g\},T\}(\varphi) &= T\{\varphi,\{f,g\}\} = T\{\{\varphi,f\},g\} - T\{\{\varphi,g\},f\}\\ &= \{g,T\}(\{\varphi,f\}) - \{f,T\}(\{\varphi,g\})\\ &= (\{f,\{g,T\}\} - \{g,\{f,T\}\})(\varphi)\end{aligned}$$

so that it is a Lie module; Leibniz identities are also easy to be verified

$$\begin{aligned}(\{f,g\}T)(\varphi) &= T(\{f,g\}\varphi) = T(\{f\varphi,g\} - \{\varphi,g\}f)\\ &= \{g,T\}(f\varphi) - (fT)(\{\varphi,g\})\\ &= (f\{g,T\} - \{g,fT\})(\varphi)\end{aligned}$$

Multiplicativity reduces to a simple computation which uses the previous one: indeed

$$\begin{aligned}\{fg,T\}(\varphi) &= T(\{\varphi,fg\}) = T(f\{\varphi,g\}) + T(g\{\varphi,f\})\\ &= \{g,fT\}(\varphi) + \{f,gT\}(\varphi)\end{aligned}$$

hence, by Leibniz identity

$$\begin{aligned}\{fg,T\} &= \{g,fT\} + \{f,gT\} = f\{g,T\} - \{f,g\}T + g\{f,T\} - \{g,f\}T\\ &= f\{g,T\} + g\{f,T\}\end{aligned}$$

thus the multiplicative identity for the Poisson module $\mathcal{D}(M)'$.

□

Notice that it also makes sense to write a “skew-symmetric” identity on $\mathcal{D}'$:

$$\{f,T\}(\varphi) + \{\varphi,T\}(f) = 0$$

Moreover notice that $\mathcal{D}'$ contains as a dense subspace, the space of distributions with compact support (cf. [4, §III-7]) which are precisely the elements of the topological dual $C^\infty(M)'$, classically denoted as $\mathcal{E}(M)'$ or simply as $\mathcal{E}'$.

If the Poisson manifold is oriented, we have both the regular distribution (cf. [4, §I])

$$T_f(\varphi) = \int_M \varphi f$$

induced by a function $f \in \mathcal{E}(M)$, and the Hamiltonian distribution

$$X_f(\varphi) = \int_M \{f, \varphi\}$$

such that

$$\{f, T_g\} + \{g, T_f\} = X_{fg}$$

as it follows from Leibniz identity for functions. Moreover

$$X_{\{f,g\}} = \{g, X_f\} - \{f, X_g\}$$

by Jacobi identity for functions, and these two distributions are related by the identity

$$\begin{aligned} T_{\{f,g\}}(\varphi) &= \int_M \{f, g\}\varphi = \int_M \{f, g\varphi\} - \int_M \{f, \varphi\}g \\ &= X_f(g\varphi) - T_g\{f, \varphi\} = (gX_f - \{f, T_g\})(\varphi) \end{aligned}$$

so that, from $T_{\{f,g\}} + T_{\{g,f\}} = 0$, it follows

$$\begin{aligned} fX_g + gX_f &= T_{\{g,f\}} + \{T_f, g\} + T_{\{f,g\}} + \{T_g, f\} = X_{fg} \\ &= \{f, T_g\} + \{g, T_f\} \end{aligned}$$

Notice that, if M is symplectic, the volume element is the top degree exterior power of the symplectic form ω (up to normalisations), so we have, by Stokes' theorem

$$\begin{aligned} X_f(\varphi) &= \int_M \{f, \varphi\}\omega^n = \int_M \operatorname{div}(\varphi X_f)\omega^n \\ &= \int_M \mathcal{L}_{\varphi X_f}\omega^n = \int_M d\mathbf{i}_{\varphi X_f}\omega^n = \int_{\partial M} \varphi \mathbf{i}_{X_f}\omega^n \end{aligned}$$

(since $d\omega = 0$ and $\mathcal{L}_{X_f}\omega = 0$ because X_f is Hamiltonian and *a fortiori* canonical). Hence

Proposition 4.2 *On a symplectic manifold (without boundary) the distribution X_f is zero.*

This result has to be interpreted as an invariance condition of the Hilbert space product on $L^2(M, \omega^n)$ w.r.t. the Poisson brackets; the non vanishing of these distributions is a measure of the "non-symplecticity" of a Poisson manifold.

The structure of Poisson module on $\mathcal{D}'$ can also be written in terms of the action of a differential operator D on distributions:

$$(DT)(\varphi) = (-1)^d T(D\varphi)$$

where d is the order of the differential operator, considering $D = X_f$ (Hamiltonian vector field corresponding to the Hamiltonian function f):

$$\{f, T\} = X_f T$$

Moreover, notice that it is impossible to define a Lie structure on $\mathcal{D}'$ for the same reasons that forbid the existence of an associative product (cf. [4, §V-1]). Now we want to introduce a remarkable submodule of $\mathcal{D}'$, whose elements we can define as follows

Definition 4.3 *A Casimir distribution on a Poisson manifold M is a distribution $T \in \mathcal{D}(M)'$ such that*

$$\forall f \in C^\infty(M) \qquad \{f, T\} = 0$$

The space of Casimir distributions will be denoted by $\mathcal{C}(M)'$.

Thus a distribution T is Casimir if $T\{\varphi, f\} = 0$ for each $f \in C^\infty(M)$ and for each $\varphi \in C_c^\infty(M)$. We have used this terminology, because of the following example: if the Poisson manifold is oriented[2] (for example if it is symplectic) then we have an immersion $T : C^\infty(M) \longrightarrow \mathcal{D}(M)'$ (with dense image) which put in correspondence the function $f \in C^\infty(M)$ with the regular distribution

$$T_f(\varphi) = \int_M f\varphi$$

In this case:

Proposition 4.4 *If $c \in \operatorname{Cas} M$ is a Casimir function then $T_c \in \mathcal{C}(M)'$.*

[2]Actually this is not strictly needed if we consider the concept of an even and odd differential form (and current): cf. [3, §1] and [4, §IX-2].

PROOF: Be $c \in \operatorname{Cas} M$: for every $f \in C^\infty(M)$ and $\varphi \in C_c^\infty(M)$ we have

$$\{f, T_c\}(\varphi) = \int_M c\{\varphi, f\} = \int_M \{\varphi, cf\}$$

We want to show that the distribution $\{f, T_c\}$ does vanish, and it suffices to do it in every open neighbourhood U (by the so called localisation principle, cf. [4, §III-8]); since we are on a Poisson manifold, we can assume the neighbourhood to be of the form $U = S \times N$ with S symplectic; therefore

$$\int_U \{\varphi, cf\} = \int_N \int_S \{\varphi|_S, cf|_S\}|_S$$

by Fubini's theorem, which we can apply since $\{\varphi, cf\} \in C_c^\infty(U)$ (we are considering Liouville measure on S); remember that the evaluation of Poisson brackets at a point x is the value of the symplectic bracket of the leaf passing through x (evalued on the restriction of the functions to the leaf itself). But

$$\int_S \{\varphi|_S, cf|_S\} = 0$$

because the integral of the image of a Hamiltonian vector field with compact support on a symplectic manifold is zero (proposition 4.0).

□

The more the Poisson structure is far from being symplectic, the more the structure of the module of Casimir distributions is non trivial: for example, in the extreme case of a Poisson manifold with the zero Poisson tensor, we have $\mathcal{C}' = \mathcal{D}'$ (every distribution is Casimir).

Proposition 4.5 *If M is symplectic then $\mathcal{C}(M)'$ is the space of locally constant distributions.*

PROOF: Being M symplectic, we have a global immersion $f \mapsto T_f$ with dense image of $C^\infty(M)$ into $\mathcal{D}(M)'$, by means of the integrations w.r.t. Liouville measure on M. Hence it makes sense to speak about locally constant distributions, as those associated to locally constant functions on M: for the sake of simplicity we will assume M to be connected.

Now pick a Casimir distribution $T \in \mathcal{C}'(M)$ we want to prove that it is of the form T_c for some constant $c \in \mathbb{R}$. Again we use Schwartz localisation principle for distributions: to prove that two distributions coincide, it suffices to check it locally; so we can suppose that $M = \mathbb{R}^{2n}$ with the canonical symplectic structure (by Darboux theorem) and coordinates $(q_1, ..., q_n, p_1, ..., p_n)$. Then

$$0 = \{f, T\}(\varphi) = \sum_{i=1}^{n} T\left(\frac{\partial\varphi}{\partial q_i}\frac{\partial f}{\partial p_i}\right) - \sum_{i=1}^{n} T\left(\frac{\partial f}{\partial q_i}\frac{\partial \varphi}{\partial p_i}\right)$$

for each $f \in C^\infty(M)$ and $\varphi \in C_c^\infty(M)$. It follows for instance that, by putting $f = q_1, ..., p_n$, for each $\varphi \in C_c^\infty(M)$:

$$T\left(\frac{\partial\varphi}{\partial p_i}\right) = T\left(\frac{\partial\varphi}{\partial q_j}\right) = 0$$

if $i, j = 1, ..., n$. But then the distribution T is constant (cf. [4, §II-6]) so that it is of the form

$$T(\varphi) = k\int\varphi$$

□

Notice that each constant gives rise to a Casimir distribution, because of proposition 4.0:

$$\int\{\varphi, f\} = 0$$

on a symplectic manifold, if $\varphi \in C_c^\infty(M)$ and $f \in C^\infty(M)$ (by integrating w.r.t. Liouville measure).

Thus, on a symplectic manifold, Casimir distributions do not give rise to anything new: this is not the case on general Poisson manifolds.

Example 4.6 *Consider once again the symplectic plane $\mathbb{R}^2_\pi$ with $\pi = x^2+y^2$, thus with brackets*

$$\{f, g\}_0(x, y) = (x^2 + y^2)\{f, g\}$$

(where $\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$ are the canonical symplectic brackets on the plane.) Of course Casimir functions are constants, just as in the symplectic case: however, while $\mathcal{C}(\mathbb{R}^2)' = \mathbb{R}$, there exist Casimir distributions which are not constant on $\mathbb{R}^2_0$: it is indeed obvious that a distribution with support in the singular point (the origin) will be a good candidate).

For example Dirac distribution δ_0 with support at the origin is Casimir, since[3]

$$\delta(\{\varphi, f\}_0) = \{\varphi, f\}_0(0) = 0$$

[3]Recall that $(DT)(\varphi) = (-1)^d T(D\varphi)$ where D is a differential operator of order d.

Its first derivatives are Casimir distributions too: for example

$$\frac{\partial\delta_0}{\partial x}(\{\varphi,f\}_0) = -\left(2x\{\varphi,f\}+(x^2+y^2)\frac{\partial\{\varphi,f\}}{\partial x}\right)\Big|_0 = 0$$

as its mixed derivative $\partial^2\delta_0/\partial x\partial y$ is. However notice that

$$\frac{\partial^2\delta_0}{\partial x^2}(\{\varphi,f\}_0) = \left(2\{\varphi,f\}+2xF(x,y)+(x^2+y^2)\frac{\partial^2\{\varphi,f\}}{\partial x^2}\right)\Big|_0 = 2\{\varphi,f\}(0)$$

which is not zero in general. So the space $\mathcal{C}(M)' = \mathbb{R}^5$ has the following generators 1, δ_0, $(\delta_0)_x$, $(\delta_0)_y$, $(\delta_0)_{xy}$.

Of course the structure of $\mathcal{C}'$ heavily depends on the Poisson tensor: if we consider on $\mathbb{R}^2$ the Poisson structure

$$\{f,g\}_\pi(x,y) = \pi(x,y)\{f,g\}$$

where $\pi \in C^\infty(\mathbb{R}^2)$ vanishes at the origin then we get a Poisson manifold with the same symplectic leaves of $\mathbb{R}^2_0$ but which may admit infinitely many independent Casimir distributions: it will suffice to consider a function in the Borel kernel, with all its derivatives vanishing at the origin to get the all space of distributions with support in the origin itself[4] is contained in $\mathcal{C}'$.

In general the study of Casimir distributions for planar Poisson structures should be connected to the classification of such structures.

Let us consider some further examples of Casimir distributions:

Theorem 4.7 *If M and N are Poisson manifold then*

$$\mathcal{C}(M\times N)' \cong \mathcal{C}(M)'\otimes\mathcal{C}(N)'$$

(topological tensor product between nuclear spaces).

PROOF: First of all notice that the statement makes sense: indeed the space $\mathcal{C}(M)'$ is a subspace of $\mathcal{D}(M)'$, hence it is a nuclear topological vector space (cf. [4, §IV-4]) and the tensor product is uniquely defined.

We work now at an algebraic level: the Poisson structure on the algebra $C^\infty(M\times N) \cong C^\infty(M)\otimes C^\infty(N)$ (cf. [5, p. 531]) is given by brackets

$$\{f_1\otimes g_1, f_2\otimes g_2\} = \{f_1,f_2\}_M\otimes g_1g_2 + f_1f_2\otimes\{g_1,g_2\}_N$$

[4] The space of such distributions is the vector space spanned by δ_0 and by all its derivatives, cf. [4, §III-10].

By Schwartz's kernel theorem (cf. [5, p. 531]), $\mathcal{D}(M\times N)' \cong \mathcal{D}(M)'\otimes\mathcal{D}(N)'$, and

$$\begin{aligned}\{T\otimes S, f\otimes g\}(\varphi\otimes\psi) &= T\otimes S\{f\otimes g, \varphi\otimes\psi\}\\ &= T\otimes S\left(\{f,\varphi\}_M\otimes g\psi + f\varphi\otimes\{g,\psi\}_N\right)\\ &= T(\{f,\varphi\}_M)S(g\psi)+T(f\varphi)S(\{g,\psi\}_N)\\ &= \left[(\{T,f\}_M\otimes gS)+(fT\otimes\{S,g\}_N)\right](\varphi\otimes\psi)\end{aligned}$$

This means that if $C\in\mathcal{C}(M)'$ and $D\in\mathcal{C}(N)'$ then $C\otimes D\in\mathcal{C}(M\times N)'$; vice versa, be $C\in\mathcal{C}(M\times N)'$: by the kernel theorem this distribution is linear combination of elements of the form $T_i\otimes S_i$, hence, for each f,g,φ,ψ (in the suitable spaces):

$$\begin{aligned}0 &= \{C, f\otimes g\}(\varphi\otimes\psi) = C\{f\otimes g,\varphi\otimes\psi\}\\ &= \textstyle\sum_i a_iT_i\{f,\varphi\}_M S_i(g\psi)+\sum_i a_iT_i(f\varphi)S_i\{g,\psi\}_N\end{aligned}$$

Functions occurring in these equations were arbitrarily chosen, so (for example by taking f to be a constant):

$$0=\sum_i a_i fT_i(\varphi)S_i\{g,\psi\}_N$$

Thus, by arbitrariness of φ, $\{S_i,g\}_N=0$. Analogously we get $\{T_i,f\}_M=0$, henceforth $C\in\mathcal{C}(M)'\otimes\mathcal{C}(N)'$.

□

Example 4.8 *If S is symplectic (and connected): $\mathcal{C}(S\times N)'\cong\mathcal{C}(N)'$; in particular, if N is a Poisson manifold endowed with the zero Poisson tensor, we get $\mathcal{C}(S\times N)'\cong\mathcal{D}(N)'$ and if N is compact $\mathcal{C}(S\times N)'\cong\mathcal{E}(N)$ is the topological dual of the space* $\mathrm{Cas}\,(S\times N)$.

When the Poisson structure is regular, a Casimir function c is, in a local neighbourhood of Darboux–Weinstein type $U=S\times N$, constant along the factor S, so that if X is a tangent vector field to S, we have $Xc=0$ (brackets on N are identically zero, since the manifold is regular); thus we can identify this distribution with a functional on $C^\infty(N)$:

Theorem 4.9 *If M is a regular Poisson manifold then $\mathcal{C}(M)'\cong\mathrm{Cas}\,(M)'$ (topological dual of the space of Casimir distributions).*

PROOF: If M is regular and $T \in \mathcal{C}(M)'$, thus $\{T, f\} = 0$ for each $f \in \mathcal{D}(M)$, then, in each local chart U, we have $\{T, f\}|_U = 0$, thus

$$0 = \{T, f\}(\varphi) = \sum_{i=1}^{r} T\left(\frac{\partial f}{\partial p_i}\frac{\partial \varphi}{\partial q_i}\right) - \sum_{i=1}^{r} T\left(\frac{\partial f}{\partial q_i}\frac{\partial \varphi}{\partial p_i}\right)$$

where $r \leq n$ is the rank of the manifold M: this follows from Weinstein's splitting theorem which, in the regular case, affirms that the Poisson structure is locally the product of a symplectic structure and of a null structure. Therefore a Casimir distribution is, in each Darboux–Weinstein local chart $U = S \times N$, a distribution of the form $k_S \otimes T_N$ where k_S is a constant, and $T_N \in \mathcal{D}(N)'$. If $c \in \mathrm{Cas}\,(M)$ then we can compute on it a functional $\widetilde{T}$ as

$$\widetilde{T}(c) = \sum_{U} T(\psi_U c) = \sum_{U=S\times N} k_S T_N(c|_N)$$

where $\{\psi_U\}$ is a partition of the unity subordinate to the covering $\{U = S \times N\}$, being $c|_N$ a function depending only on the coordinates of the N factor. Hence we have a map $\widetilde{\ }: \mathcal{C}(M)' \longrightarrow \mathrm{Cas}\,(M)'$ which is injective since $\widetilde{T} = 0$ if and only if each T_N is zero, so that T is zero, and surjective since a functional $\gamma \in \mathrm{Cas}\,(M)'$ is induced by a distribution which locally is defined as $T_U = 1 \otimes \gamma|_U$ (the restriction $\gamma|_U$ defines a distribution on N since $\mathrm{Cas}\,(U) = C^\infty(N)$).

□

Example 4.10 *Consider the Poisson manifold $M = \mathfrak{so}(3)^* \setminus \{0\}$, thus the Lie–Poisson manifold associated to the Lie algebra $\mathfrak{so}(3)$ minus the origin; it is a regular Poisson manifold whose leaves are concentric spheres S_r centered at the origin and with positive radii r. A Casimir distribution is a distribution $T \in \mathcal{D}(\mathbb{R}^3 \setminus 0)$ such that*

$$0 = T\{f, \varphi\} = T(\nabla f \wedge \nabla\varphi)$$

where ∇ denotes the gradient and $\wedge$ the vector product. More precisely, in Cartesian coordinates:

$$T\{f, \varphi\} = T(x(f_y\varphi_z - f_z\varphi_y) + y(f_z\varphi_x - f_x\varphi_z) + z(f_x\varphi_y - f_y\varphi_x))$$

($f_x = \frac{\partial f}{\partial x}$ and so on) so that the condition $T\{f,\varphi\} = 0$ implies

$$\begin{cases} x\dfrac{\partial T}{\partial y} = y\dfrac{\partial T}{\partial x} \\ y\dfrac{\partial T}{\partial z} = z\dfrac{\partial T}{\partial y} \\ z\dfrac{\partial T}{\partial x} = x\dfrac{\partial T}{\partial z} \end{cases}$$

But since $(x,y,z) \neq 0$, a Casimir distribution is determined whenever one of its non identically zero partial derivatives is given (if all its partial derivatives are zero then the distribution would be the integral by a constant function). This means that the space of Casimir distributions is the space of distributions whose derivatives along the directions tangent to the leaves are zero: thus they will be continuous linear functionals on a one dimensional space (depending on a parameter which is exactly the distance from the origin in polar coordinates on $\mathbb{R}^3$), and which, in fact, may be identified with $C^\infty(\mathbb{R}_+)'$, the dual of the space of Casimir functions, in agreement with the previous theorem.

Notice that, if we consider the all Lie–Poisson manifold $\mathfrak{so}(3)^*$ we would have at least a distribution not induced by any linear functional on the space of Casimir functions: Dirac measure concentrated at the origin (which is the singular point of the Poisson manifold). Therefore, in this case, each Casimir function induces a Casimir distribution but not vice versa, as we expect from the theorem.

In general, a distribution with support in a point with null rank is Casimir; for example consider the Poisson structure in the plane $\mathbb{R}^2_\pi$ with brackets

$$\{f,g\}(x,y) = \pi(x,y)\{f,g\}_S(x,y)$$

where $\{\}_S$ are the usual symplectic brackets and π a smooth function: then a Casimir distribution T is such that

$$\forall f \in \mathcal{E}(\mathbb{R}^2)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^2) \quad 0 = T(\pi\{f,\varphi\}_S) = (\pi T)\{f,\varphi\}_S$$

so that πT is constant, This is the case, for instance, if $\operatorname{supp}\pi \cap \operatorname{supp} T = \emptyset$, as, in particular, for $T = \delta_0$ and $\pi(0,0) = 0$; notice that not every distribution with support in the origin is Casimir, unless π belongs to the Borel kernel (all its derivatives are zero in that point).

Example 4.11 *Consider $\mathbb{R}^2$ with brackets induced by the function $\pi = y^2$: then not only Dirac's measures concentrated at the singular points (thus the ones on the line $\{y = 0\}$) are Casimir distributions (as their derivatives w.r.t. the x of any order), but if we consider every distribution $R \in \mathcal{D}(\mathbb{R})'$ on the line $\{y = 0\}$ and we extend it to $\mathbb{R}^2$ as $T(\varphi) = R(\varphi \circ i)$ (with $i(x) = (x, 0)$) then, of course, $T \in \mathcal{C}(\mathbb{R}^2)'$, so that we have an inclusion $\mathcal{D}(\mathbb{R})' \longrightarrow \mathcal{C}(\mathbb{R}^2)'$.*

Example 4.12 *A Lie–Poisson manifold $\mathfrak{g}^*$ has at least a point of rank zero: the origin; thus it admits Casimir distributions with support in $\{0\}$. If we write in coordinates the Lie–Poisson structure as*

$$\pi = \sum_{i,j} \sum_k c_{ij}^k x_k \partial_i \wedge \partial_j$$

then a distribution T is Casimir if and only if

$$\sum_{i,j,k} T(c_{ij}^k x_k \partial_i f \partial_j \varphi) = 0$$

for each $f \in C^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Hence

$$0 = \sum_{i,j,k} c_{ij}^k x_k T(\partial_i f \partial_j \varphi)$$

Of course δ_0 is Casimir, while its first derivatives are not:

$$\partial_h \delta_0 \{f, \varphi\} = \partial_h \{f, \varphi\}(0) = \sum_{i,j} c_{ij}^h \partial_i f(0) \partial_j \varphi(0)$$

nor, a fortiori, higher ones.

5 A sketch of integration theory

We have developed a general setting for differential Poisson calculus [1]: here we try to sketch accordingly some integral calculus on Poisson manifolds.

Consider the decomposition in symplectic leaves $\bigcup_{x \in M} S_x$ of a Poisson manifold M: if $c \subset S_x$ is a p-chain (cf. [3, §6]) in the symplectic leaf through $x \in M$, and $P \in \mathfrak{S}_c^p(M)$ is a symplectic tensor field with compact support on M, then we can consider the restriction $P|_{S_x}$ which is a skew-field on S_x and, via the symplectic form ω on S_x induced by the Poisson structure on

M, we can identify it with a compactly supported p-form $\omega^\flat(P|_S)$, where $\omega^\flat : TS_x \longrightarrow TS_x^*$ is the isomorphism associated to the non-degenerate form ω, and integrate it over c, by defining

$$\int_c P = \int_c \omega^\flat(P|_S)$$

Notice that, if $c \subset S$ is a symplectic chain and $P \in \mathfrak{S}_c^p(M)$ then $P = \pi^\# \alpha$ (with $\alpha \in \Omega^p(M)$ and $\langle \pi^\# \alpha, \beta_1 \wedge ... \wedge \beta_n \rangle = (-1)^n \langle \alpha, \pi^\# \beta_1 \wedge ... \wedge \pi^\# \beta_n \rangle$, cf. [6, pag. 43]) and, by commutativity of the previous diagram we have $\omega^\#(i_S^* \alpha) = (\pi^\# \alpha)|_S$, so that

$$\int_c P = \int_c i_S^* \alpha$$

where $i_S : S \longrightarrow M$ is the inclusion.

We can extend this definition to linear combinations of chains lying in different leaves: if $c_1 \subset S_{x_1}$,...,$c_k \subset S_{x_k}$ and $P \in \mathfrak{S}_c^p(M)$ we simply define

$$\int_{\sum_i a_i c_i} P = \sum_i a_i \int_{c_i} P$$

Definition 5.1 *A p-chain $\sum_i a_i c_i$ such that c_i is contained in a single symplectic leaf is called symplectic chain.*

For example, if $X \in \mathfrak{X}_c^1(M)$ is a Hamiltonian vector field with compact support, then $X = \pi^\# d\varphi$ for some $\varphi \in C_c^\infty(M)$, and if S is a symplectic leaf, then, for a 1-chain c contained in S:

$$\int_c X = \int_c i_S^* d\varphi = \int_c d i_S^* \varphi = \int_{\partial c} i_S^* \varphi = \varphi(x_1) - \varphi(x_0)$$

where $\partial c = x_0 + x_1$ denotes the boundary of the chain c.

In particular we can consider integration over symplectic leaves if $P \in \mathfrak{S}_c^p(M)$ is a symplectic tensor field with compact support in each leaf, for example if leaves are compact (this always makes sense because symplectic leaves are oriented manifolds).

Symplectic chains with the boundary operator form a singular complex $\mathcal{S}^S(M)$ which is a subcomplex of the usual singular complex $\mathcal{S}(M)$ of M, and which induces some singular homology groups $H_k^S(M)$. Since a symplectic cycle is also a cycle in M, we have a map

$$H_k(M) \longrightarrow H_k^S(M)$$

Example 5.2 *In $\mathbb{R}^2$ with the Poisson structure induced by a smooth function $\pi(x, y)$, a symplectic 0-chain is a linear combination of points in which the Poisson tensor identically vanishes or is never zero, and a 1-chain or a 2-chain is a linear combination of 1-simplexes or 2-simplexes σ_i such that $\pi \circ \sigma_i$ never vanish. For instance, if $\pi = x^2 + y^2$ then we have*

$$H_0^S(\mathbb{R}_0^2) = \mathbb{R} \qquad H_1^S(\mathbb{R}_0^2) = \mathbb{R} \qquad H_2^S(\mathbb{R}_0^2) = 0$$

If $\pi = y^2$ then

$$H_0^S(\mathbb{R}_0^2) = \mathbb{R}^{\mathbb{R}} \qquad H_1^S(\mathbb{R}_0^2) = 0 \qquad H_2^S(\mathbb{R}_0^2) = 0$$

Notice that in this case the 0-th homology space is infinite dimensional since it simply counts the number of leaves: more generally we can remark that

Proposition 5.3 *If M is a Poisson manifold then homology groups $H_i^S(M)$ coincide with the singular homology groups of the topological space M w.r.t. the leaf-topology.*

It suffices to remember that the leaf topology on a foliated space is the topology whose open sets are the intersection between open sets in the topology of the manifold and leaves[5].

Of course, if M is a symplectic manifold, these concepts reduce to the usual notions of singular chain, usual integration and singular homology.

Symplectic chains are examples of more general objects which can be also considered as generalisations of distributions, and which we will call *currents*: actually we will not deal with currents in the de Rham sense but with some kind of linear functionals which are the analogous, in the Poisson case, of the classical de Rham currents (cf. [3, §3]).

Definition 5.4 *A p-Poisson current on a Poisson manifold M is a continuous linear functional on the Fréchet space $\mathfrak{S}_c^p(M)$ of symplectic tensors of order p and with compact support in M; p is the order of the current. We denote the space of p-Poisson currents with the symbol $\mathcal{X}'_p$ and with $\mathcal{X}'$ the space of Poisson currents of arbitrary order.*

[5] Notice that, while in this topology the space M remains paracompact, is no longer a pure manifold (in the sense of Bourbaki), i.e. its dimension may vary (being the rank of the Poisson structure), but in every case we can consider the singular homology groups w.r.t. this leaf topology, and, since leaves are the connected components, it is obvious that we obtain the groups so far defined.

Of course a 0-current is nothing else than a distribution. Moreover, if c is a symplectic p-chain then the map

$$P \mapsto \int_c P$$

defines a p-current, where $P \in \mathfrak{S}_c(M)$; also if α is a p-form on M then it induces the p-Poisson current

$$T_\alpha(P) = \int_M \mathbf{i}_P \alpha$$

(by contracting α over P).

These examples are not surprising, since a Poisson structure defines a map $\pi^\# : T^*M \longrightarrow TM$ and if T is a Poisson current then it induces a de Rham current D_T in the following way: for each form α

$$D_T(\alpha) = T(\pi^\# \alpha)$$

Since $\mathrm{Im}\, \pi^\# = \mathfrak{S}$, this defines a current for each form α. Of course, if M is symplectic then $\pi^\#$ is an isomorphism which induces an isomorphism between the space of Poisson currents and the space of de Rham currents.

If M is the null Poisson manifold then $\pi^\# = 0$ and so there is only one Poisson current: the zero one.

Now we want to set up a complex with Poisson currents, and, to do this, we need a boundary operator: since symplectic chains are currents this operator has to restrict to the boundary of chains. We draw inspiration from Stokes theorem, which, if c is a 1-chain and X_f a Hamiltonian vector field, reads as

$$\int_c X_f = \int_{\partial c} f$$

But the operator $X : C_c^\infty(M) \longrightarrow \mathrm{Ham}_c(M)$ is the Schouten multiplication by the Poisson tensor π, which in general defines the coboundary operator of the Poisson complex $(\mathfrak{X}^{\wedge n}(M), d_\pi)$:

$$d_\pi T = -[\![\pi, T]\!]$$

where $[\![,]\!]$ here denotes Schouten brackets (cf. [6, §1, §5]).

Lemma 5.5 *If $P \in \mathfrak{S}_c^p(M)$ and $Q \in \mathfrak{S}_c^q(M)$ then $[\![P, Q]\!] \in \mathfrak{S}_c^{p+q-1}(M)$.*

PROOF: We use Nijenhuis formula for Schouten brackets:

$$\mathbf{i}_{[\![P,Q]\!]}\alpha = (-1)^{q(p+1)}\mathbf{i}_P d\mathbf{i}_Q\alpha + (-1)^p \mathbf{i}_Q d\mathbf{i}_P\alpha - \mathbf{i}_{P\wedge Q}\alpha$$

(cf. [B-V] Theor. 2.21).

Now, by hypothesis and lemma 2.0., $P \in \mathfrak{S}_c^p(M)$ and $Q \in \mathfrak{S}_c^q(M)$ are such that, if $i_S^*\alpha = 0$ then $\mathbf{i}_P\alpha = \mathbf{i}_Q\alpha = 0$ (where $i_S : S \longrightarrow M$ is the inclusion of an arbitrary symplectic leaf). We have to check in this case that also $\mathbf{i}_{[\![P,Q]\!]}\alpha = 0$, which is trivial: $\mathbf{i}_P d\mathbf{i}_Q\alpha = 0$ and $\mathbf{i}_Q d\mathbf{i}_P\alpha = 0$, so that

$$\mathbf{i}_{P\wedge Q}\alpha = \mathbf{i}_P\mathbf{i}_Q\alpha = 0$$

whence, by Nijenhuis formula, $\mathbf{i}_{[\![P,Q]\!]}\alpha = 0$.

□

In particular, $[\![\pi, T]\!]$ is again a symplectic tensor, and we can define the boundary of a Poisson current $T \in \mathcal{X}'_p$ as

$$bT(P) = -T([\![\pi, P]\!])$$

if $P \in \mathfrak{S}_c^p(M)$.

Notice that $b : \mathcal{X}'_p \longrightarrow \mathcal{X}'_{p-1}$, and that this is a boundary operator since d_π is a coboundary one (because of the graded Jacobi identity for Schouten brackets):

$$bbT(P) = -bT([\![\pi, P]\!]) = T([\![\pi, [\![\pi, P]\!]]\!]) = 0$$

We can consider homology groups for this complex which we will denote $H_*^{Sp}(M)$ and call *symplectic homology groups.*

For instance, since in a symplectic manifold $\mathfrak{S}(S) \cong \Omega(S)$ we get that this homology is precisely de Rham homology (cf. [3, §4]) in this case:

Theorem 5.6 *If S is symplectic then $H^{Sp}(M) = H^{dR}(M)$ (de Rham homology space).*

Moreover, since a 0-current is a distribution, and in this case $bT(\varphi) = T(X_\varphi)$ we find that $H^0(M) = \mathcal{C}(M)'$ is the space of Casimir distributions.

Of course if M is a null Poisson manifold then its homology (from degree one on) is trivial, being trivial its complex; in general the map D which sends a Poisson current into a de Rham current induces a map $D_* : H^{Sp}(M) \longrightarrow H^{dR}(M)$ in homology: in fact

$$bD_T(\alpha) = D_T(d\alpha) = T(\pi^\# d\alpha) = T(-[\![\pi, \pi^\#\alpha]\!]) = bT(\pi^\#\alpha) = D_{bT}(\alpha)$$

since $\pi^{\#}d\alpha + d_\pi\pi^{\#}\alpha = 0$ (cf. [6, pag. 43]).

The homology we defined above contains the singular homology of symplectic chains; but notice that we may also define a cohomology, using symplectic fields: in fact we can consider the complex

$$0 \longrightarrow C^\infty(M) \longrightarrow \mathfrak{S}(M) \longrightarrow \mathfrak{S}^2(M) \longrightarrow \mathfrak{S}^3(M) \longrightarrow ...$$

with the operator $d_\pi(P) = -[\![\pi, P]\!]$; as we know this operator is a coboundary, so that we have a cohomology $H^*_S(M)$ ring: in fact wedge product of symplectic tensors induces a morphism in cohomology, since (cf. [B-V] Prop. 2.16)

$$d_\pi(P \wedge Q) = (d_\pi P) \wedge Q + (-1)^p P \wedge d_\pi Q$$

We call this cohomology *symplectic cohomology,* and notice that we have introduced it algebraically in [1] as the cohomology of the Lie algebra $\mathcal{H}_A$.

We remark explicitly that this cohomology is not the usual Poisson cohomology of the manifold (cf. [6, §5]), since our complex is a subcomplex of the usual Poisson complex, which is formed by all skew-symmetric contravariant tensor fields. If we denote by $H_{LP}(M)$ the usual Poisson cohomology, we have that

Theorem 5.7 *There is a ring morphism $H^*_S(M) \longrightarrow H_{LP}(M)$.*

(of course $H_{LP}(M)$ has a ring structure induced by the wedge product between symplectic tensors.)

Of course, if the manifold is symplectic then $\mathfrak{S}(M) = \mathfrak{X}(M)$ which is, via $\pi^{\#}$, isomorphic to $\Omega^1(M)$, so that all these cohomologies coincide with the de Rham one.

On the other hand, if M is the null Poisson manifold then $\mathfrak{S}(M) = 0$ so that $H^0_S(M) = C^\infty(M)$ but $H^k_S(M) = 0$ if $k > 0$, while the cohomology groups w.r.t. the Poisson cohomology are the spaces of tensor fields themselves $\mathfrak{X}^k(M)$; this shows that symplectic cohomology is coarser than Poisson cohomology: indeed it is well known (cf. [6, §5.1]) that $H^1_{LP}(M)$ coincides with $\mathrm{Can}\,(M)/\mathrm{Ham}\,(M)$, while $H^1_S(M)$ is $(\mathrm{Can}\,(M) \cap \mathfrak{S}(M))/\mathrm{Ham}\,(M)$ (by the same computation): so, for example, when M has the null Poisson structure we find, as just stated, $H^1_{LP}(M) = \mathfrak{X}(M)$, while $H^1_S(M) = 0$.

Moreover symplectic cohomology is a bit more functorial than Poisson one: indeed if $F : M \longrightarrow N$ is a smooth Poisson submersion between Poisson manifold, thus a smooth submersion such that

$$F^*\{f, g\}_M = \{F^*f, F^*g\}_N$$

then we can pull-back symplectic tensors from N to M: this is done firstly on Hamiltonian vector fields as

$$F^*X_f := X_{F^*f}$$

and on any symplectic field by $C^\infty(N)$-linearity:

$$F^*(fX_g) := (F^*f)X_{F^*g}$$

Proposition 5.8 *The map $F^* : \mathfrak{S}(N) \longrightarrow \mathfrak{S}(M)$ is a well-defined morphism of differential modules.*

PROOF: If $X \in \mathfrak{S}(N) = \operatorname{Im} \pi_N^\#$ then

$$F^*X = \pi_M^\# F^*\omega$$

where $\omega \in \Omega(N)$ is any 1-form on N such that $\pi_N^\#\omega = X$: we need to show that the value of F^*X does not depend on ω but only on its image in $\mathfrak{X}(N)$; to check that, we pick a form $\gamma \in \ker \pi_N^\#$, and verify that

$$\pi_M^\# F^*\gamma = 0$$

We have to show that a vector field is zero: let us compute it on a smooth function $f \in C^\infty(M)$:

$$\pi_M^\# F^*\gamma(f) = \langle \pi_M^\# F^*\gamma, df\rangle = -\langle F^*\gamma, \pi_M^\# df\rangle = -\langle \gamma, dF(\pi_M^\# df)\rangle$$

But F is both a Poisson map, so that $dF\pi_M^\# F^* = \pi_N^\#$ (cf. [6, §7.1]), and a submersion, so that $f = F^*\varphi$ for some $\varphi \in C^\infty(N)$, therefore

$$\begin{aligned}
\pi_M^\# F^*\gamma(f) &= -\langle \gamma, dF(\pi_M^\# df)\rangle = -\langle \gamma, dF(\pi_M^\# dF^*\varphi)\rangle \\
&= -\langle \gamma, dF(\pi_M^\# F^* d\varphi)\rangle = -\langle \gamma, \pi_N^\# d\varphi\rangle \\
&= \langle \pi_N^\#\gamma, d\varphi\rangle = 0
\end{aligned}$$

(since $\gamma \in \ker \pi_N^\#$.)

Now we show that F^* is a morphism of modules: this follows from the fact that F it is a Poisson map:

$$\begin{aligned}
F^*(fX) &= F^*\left(f \textstyle\sum_i f_i X_{h_i}\right) = \textstyle\sum_i F^*(ff_i)X_{F^*h_i} \\
&= \textstyle\sum_i F^*(f)F^*(f_i)X_{F^*h_i} = F^*(f)F^*(X)
\end{aligned}$$

Moreover, $F^*X_f = X_{F^*f}$ means exactly that the diagram

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{F^*} & C^\infty(M) \\ \downarrow d_\pi & & d_\pi \downarrow \\ \mathfrak{S}(N) & \xrightarrow{F^*} & \mathfrak{S}(M) \end{array}$$

does commute.

□

Of course we call the morphism of modules $F^* : \mathfrak{S}(N) \longrightarrow \mathfrak{S}(M)$ induced by a Poisson map the *pull-back* map. Suppose that $X = \sum_i f_i X_{h_i}$ with $f_i, h_i \in C^\infty(N)$: then

$$dF \circ F^*X = \sum_i dF \circ f_i X_{h_i} = \sum_i (f_i \circ F) F^* X_{h_i} = \sum_i (f_i \circ F) X_{F^*h_i} = X \circ F$$

so that F^* behaves really as a pull-back:

$$dF \circ F^*X = X \circ F$$

Of course we extend it to a map of DG-algebras $F^* : \mathfrak{S}^\bullet(N) \longrightarrow \mathfrak{S}^\bullet(M)$ as

$$F^*(P \wedge Q) := F^*P \wedge F^*Q$$

Corollary 5.9 *If P is a symplectic tensor on a Poisson manifold N and if $F : M \longrightarrow N$ is a Poisson map then there exists a unique symplectic pull-back F^*P on M.*

Finally we come to the promised functoriality:

Theorem 5.10 *If $F : M \longrightarrow N$ and $G : N \longrightarrow P$ are smooth Poisson submersions between Poisson manifolds then*
(1) $(G \circ F)^* = F^* \circ G^*$.
(2) *If* $F = \mathrm{Id} : M \longrightarrow M$ *then* $F^* = \mathrm{Id} : \mathfrak{S}(M) \longrightarrow \mathfrak{S}(M)$.
(3) *If* $P \in \mathfrak{S}^p(N)$ *and* $Q \in \mathfrak{S}^q(N)$ *then* $F^*[\![P, Q]\!] = [\![F^*P, F^*Q]\!]$.

PROOF:
(1) is trivial on Hamiltonian fields,

$$(GF)^*X_h = X_{(GF)^*h} = X_{F^*G^*h} = F^*X_{G^*h} = F^*G^*X_h$$

and extends linearly on symplectic fields

$$(GF)^*(fX_h) = (GF)^*f(GF)^*X_h = (F^*G^*f)(F^*G^*X_h) = F^*G^*(fX_h)$$

(2) If $F = \mathrm{Id}$ then $F^*X_h = X_h$ and $F^*(fX_h) = fX_h$ so that $F^* = \mathrm{Id}$ too: in any case the extension to arbitrary symplectic tensors follows from the definition.
(3) For $p = 1$ and $q = 0$ we have

$$F^*[gX_h, f] = F^*(gX_hf) = F^*(g\{f,h\}) = F^*g\{F^*f, F^*h\} = [F^*(gX_h), F^*f]$$

The case $p = q = 1$ is also simple: for Hamiltonian fields

$$\begin{aligned} F^*[X_f, X_g] &= F^*X_{\{f,g\}} = X_{F^*\{f,g\}} \\ &= X_{\{F^*f, F^*g\}} = [X_{F^*f}, X_{F^*g}] = [F^*X_f, F^*X_g] \end{aligned}$$

Furthermore

$$\begin{aligned} F^*[fX_h, X_g] &= F^*(f[X_h, X_g] + \{f,g\}X_h) \\ &= F^*f[F^*X_h, F^*X_g] + F^*\{f,g\}F^*X_h \\ &= F^*f[F^*X_h, F^*X_g] + \{F^*f, F^*g\}F^*X_h \\ &= [F^*fF^*X_h, F^*X_g] \end{aligned}$$

(similarly for the other variable) so that (3) holds when $p = q = 1$. Now proceed by induction on q in the case $p = 1$: recall that

$$[\![P, R \wedge X]\!] = [\![P, R]\!] \wedge X + (-1)^{r(p+1)} R \wedge [\![P, X]\!]$$

and write Q as $R \wedge X$ (for the sake of simplicity assume only one summand) with $X \in \mathfrak{S}^1(N)$; then, by induction:

$$\begin{aligned} F^*[P, Q] &= F^*[P, R] \wedge F^*X + (-1)^{r(p+1)} F^*R \wedge F^*[P, X] \\ &= [F^*P, F^*R] \wedge F^*X + (-1)^{r(p+1)} F^*R \wedge [F^*P, F^*X] \\ &= [F^*P, F^*R \wedge F^*X] = [F^*P, F^*Q] \end{aligned}$$

Notice that this works for any p, so the theorem is proven.

□

Example 5.11 *If $F : M \longrightarrow N$ is Poisson then $F^*\pi^N = \pi^M$: indeed*

$$\pi^{N\#}_{F(x)} = (dF)_x \pi^{M\#}_x F^*_{F(x)}$$

$F^*X_f = X_{F^*f}$ is a particular case of

Lemma 5.12 *The pull-back commutes with the Poisson differential.*

PROOF: By definition of d_π, if $P \in \mathfrak{S}^p(N)$:

$$F^*d_{\pi^N}P = -F^*[\pi^N, P] = -[F^*\pi^N, F^*P] = -[\pi^M, F^*P] = d_{\pi^M}F^*P$$

by the previous theorem and the example.

□

This is the result we are interested in:

Theorem 5.13 *A smooth Poisson submersion $F : M \longrightarrow N$ induces a morphism $F^* : H_S(N) \longrightarrow H_S(M)$.*

PROOF: Of course we put $F^*[P] = [F^*P]$ using the pull-back: this makes sense by the lemma, and induces a morphism since, by definition, $F^*(P\wedge Q) = F^*P \wedge F^*Q$.

□

Remember that we have defined a homology by means of singular homology of symplectic chains: now, if P is a symplectic p-tensor, then define the pairing $\langle , \rangle : \mathfrak{S}^p_c(M) \times \mathcal{S}^S(M) \longrightarrow \mathbb{R}$ as

$$\langle P, c\rangle = \int_c P$$

and notice that

$$\begin{aligned} \langle d_\pi P, c\rangle &= \int_c d_\pi P = \int_c d_\pi \pi^\# \alpha = -\int_c \pi^\# d\alpha \\ &= -\int_c i_S^* d\alpha = -\int_{\partial c} i_S^*\alpha = -\langle P, \partial c\rangle \end{aligned}$$

so that this is a skew-symmetric pairing $B : H^S_k(M) \times H^S_k(M) \longrightarrow \mathbb{R}$ between leaf singular homology and symplectic cohomology which induces a map $B^\flat : H^k_S(M) \longrightarrow H^k(M)$ as

$$B([P])([1]) = \langle P, c\rangle$$

if $B([P]) = 0$ then, for each leaf S and each chain $c \subset S$:

$$\int_c i_S^* \alpha = 0$$

where $P = \pi^\# \alpha$, so that $i_S^* \alpha = 0$ and this implies $P = 0$; *a fortiori* $[P] = 0$ so that B is injective.

Notice that this duality exists only for symplectic cohomology and leaf singular homology, since it makes no sense for general Poisson cohomology nor for the singular homology of the all manifold M: nevertheless it is not a Poincaré duality, since it can fail to be non degenerate.

Example 5.14 *Consider again $\mathbb{R}^2$ with the structure $\pi = x^2 + y^2$: the first symplectic homology group is $\mathbb{R}$, generated by a loop around the origin; for example the unit circle is certainly a cycle which is non trivial in homology. We have also an immediate cohomology class in $H_S^1(\mathbb{R}^2)$ given by the tangent vector field to this circle: $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, which is symplectic since it is zero at the origin; moreover it gives rise to a cohomology class, since $\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(x^2 + y^2) = 0$, which is not zero: indeed if, for some f:*

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = X_f$$

then

$$\begin{cases} x = (x^2 + y^2)\dfrac{\partial f}{\partial x} \\ y = (x^2 + y^2)\dfrac{\partial f}{\partial y} \end{cases}$$

so that $f = \frac{1}{2}\ln(x^2 + y^2)$ would not exists at 0; hence the field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ can't be Hamiltonian, hence it gives rise to a non trivial cohomology class.

But also the field $y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}$ is symplectic and its zero on the function $x^2 + y^2$; moreover if

$$y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} = X_f$$

then

$$\begin{cases} x = -(x^2 + y^2)\dfrac{\partial f}{\partial y} \\ y = (x^2 + y^2)\dfrac{\partial f}{\partial x} \end{cases}$$

and again $f = \arctan\frac{x}{y}$ would not globally exists; hence the field $y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}$ isn't Hamiltonian too, so that it gives rise to another non trivial cohomology *class. These classes are different, since if they belong to the same class then for some* f

$$X_f = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} = (x+y)\frac{\partial}{\partial y} - (y-x)\frac{\partial}{\partial x}$$

therefore $1 = f_{xy} = f_{yx} = -1$ *and* f *can't exist in* $\mathbb{R}^2$*. We conclude that* $H^1_S(\mathbb{R}^2) \neq \mathbb{R} = H^S_1(\mathbb{R}^2)$.

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