Poisson and Foliated Complex Structures

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Abstract. Regular Poisson structures and foliated (almost) complex structures are considered on manifolds, as a generalization of the Kähler case. We discuss some examples and make some remarks on the general case.

1 Introduction

Symplectic structures and (almost) complex structures on manifolds are interesting not only in themselves, but especially when they can be suitably combined. In particular, it is well known that, given a symplectic form ω on a manifold M, the space of compatible almost complex structures is not empty (cf. [1]), where "compatible" means

$$\begin{cases} \omega(JX, JY) = \omega(X, Y) \\ \omega(JX, X) > 0 \qquad\qquad (X \neq 0) \end{cases}$$

On the other hand, almost complex structures can be exhibited (locally and globally) such that there are no compatible symplectic forms (cf. [9], [11]). Furthermore it is nowadays a classic result that there are symplectic manifolds with no Kähler structures (see e.g. [12], [3] or, for a more detailed account, [10]).

Many classes of manifolds do not support symplectic structures: however there is a natural generalization of the concept of a symplectic structure, namely the notion of *Poisson structure,* which may arise when symplectic structures lack (see e.g. [7], [14], and, for a comprehensive exposition, [13]).

It is therefore natural to ask about a possible interplay between these "generalized symplectic structures" and (almost) complex structures. In this paper we define such a compatibility for *regular Poisson structures* not for general ones: indeed, the formers both induce foliations[1] and are G-structures[2]. We will exploit several non trivial examples, whose study is our main motivation.

In section 2 we recall the concept of a regular Poisson structure. In section 3 we discuss in details some example of regular Poisson structures on manifolds with Kähler leaves, with emphasis on a case where the manifold cannot be KählerİIn the last section we state the definition of (almost) Poisson–Kähler structure and discuss the existence of compatible almost complex structures with a given regular Poisson structure on a manifold and, vice versa, we give necessary conditions for the existence of regular Poisson structures compatible with a given almost complex one, along the same lines as in the symplectic case. Finally we give a family of foliated (almost) complex structures on the torus $\mathbb{T}^7$ that cannot be Poisson–Kähler.

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2 Regular Poisson structures as G-structures

A Poisson manifold (M, π) is a smooth manifold M equipped with a 2-contravariant skew-symmetric tensor $\pi \in \bigwedge^2 T^*M$ and such that Poisson brackets

$$\{f, g\} = \pi(df \wedge dg)$$

turns $C^\infty(M)$ into a real Lie algebra (cf. [7] or [13]). If the rank of π is constant then the Poisson manifold is said to be *regular*.

A regular Poisson manifold is naturally foliated, since the image $H(M)$ of the map $\pi^\# : T^*M \to TM$, induced by π as $i_{\pi^\#(\alpha)}(\beta) = \pi(\alpha \wedge \beta)$, is a Lie subalgebra of TM (due to Jacobi identity for Poisson brackets), hence an integrable distribution. Moreover, the restriction of π on each leaf L is of maximal rank (the dimension of each leaf is the rank of the Poisson tensor) so that it induces a symplectic structure $(\pi|_L)^{-1}$ on each leaf L. Thus a regular Poisson manifold is foliated by symplectic leaves.

[1] In general a Poisson manifold admits a generalized foliation, in a sense which may be make precise, cf. e.g. [8] or [13].

[2] This is no longer true in the general case.

The classic example is that of a symplectic manifold: it is a regular Poisson manifold whose Poisson tensor π has maximal rank, so that the map $\pi^{\#}$ is an isomorphism and the 2-form $\omega(X,Y) = \pi(\pi^{\#-1}(X), \pi^{\#-1}(Y))$ is a symplectic form on M. Its symplectic leaves are just its connected components. By contrast, every manifold is regular Poisson w.r.t. the *null Poisson structure* given by the 0 tensor: in a natural way, the product $S \times N$ of a symplectic manifold times a null Poisson manifold is equipped with a regular Poisson structure of rank $\dim S$, and Lichnerowicz *splitting theorem* (indeed foreseen by Sophus Lie) asserts that every regular Poisson manifold is locally of the form $S \times N$ being $\dim S$ its rank[3].

Recall (cf. [4]) that, if $(M, \mathcal{F})$ is a foliated manifold, one can define the foliated de Rham complex

$$\Omega^k(M, \mathcal{F}) := \{\alpha \in \Omega^k(M) \,|\, \forall x \in M \quad \alpha_x \in \bigwedge{}^k T_x^* S_x\}$$

where $\Omega^k(M)$ denotes the space of k-forms on M and S_x is the leaf passing through x. These spaces are sub-$C^\infty(M)$-modules of $\Omega^k(M)$ and, moreover, the graded algebra $\Omega^*(M, \mathcal{F}) = \bigoplus_k \Omega^k(M, \mathcal{F})$ is an ideal in $\Omega^*(M)$ w.r.t. the wedge product; furthermore, $d\Omega^k(M, \mathcal{F}) \subset \Omega^{k+1}(M, \mathcal{F})$ where d denotes the usual exterior differential on the de Rham complex.

Now we can interpret a Poisson structure in terms of foliated forms as follows [7], [4]:

Proposition 2.1 *On a manifold M a regular Poisson structures π of rank r is the same as a pairs $(\mathcal{F}, \omega)$, where $\mathcal{F}$ is a foliation and ω a foliated closed 2-form with rank r.*

An important class of regular Poisson structures which we are going to use is given by Dirac brackets[4] (cf. [7], [13], [8]).

Definition 2.2 *A* Dirac manifold *is a triple $(M, \mathcal{F}, \omega)$ where M is a manifold, $\mathcal{F}$ a foliation on M and ω a non degenerate two-form on M such that its pull-backs via the injections of any leaf into the manifold M are symplectic on the leaf.*

We are interested in Dirac manifolds since they are regular Poisson manifold, their Poisson brackets being defined as

$$\{f, g\}(x) = \{f, g\}_{L_x}(x)$$

[3]This is a particular case of Weinstein's splitting therem (cf. [13], [14]), which holds in any Poisson manifold.

[4]This notion emerges in the broader context of algebroids: we will use only its particular version we need, on manifolds, following [13].

where $\{,\}_{L_x}$ is the bracket induced by the symplectic form ω on the leaf L_x through x (this is indeed a smooth function because the distribution $x \mapsto L_x$ is smooth).

As proved by I. Vaisman (cf. [13]) every regular Poisson manifold can be embedded, as a Poisson submanifold, into a Poisson manifold whose brackets are Dirac.

3 Some Examples

In many cases symplectic leaves of a Poisson manifolds are Kähler: for example a product $S \times N$ of a Kähler manifold S times a null manifold N equipped with the Poisson product structure, or the vector space $\mathfrak{g}^*$ dual to a semisimple Lie algebra $\mathfrak{g}$ equipped with Lie–Poisson brackets (see [13] or [8] for Lie–Poisson structures: their symplectic leaves, being symplectic homogenous spaces are Kähler, cf. e.g. [1], [10]).

Example 3.1 *Consider $\mathbb{R}^3$ (with global coordinates (x, y, z)) equipped with the contact form $\alpha = dz - ydx$, and take $M = \mathbb{R}^3 \times S^1$ with the non degenerate 2-form (by ϑ we denote the local coordinate on S^1)*

$$\omega = d\alpha + \alpha \wedge d\vartheta = dx \wedge dy + dz \wedge d\vartheta - ydx \wedge d\vartheta$$

Now we build Dirac brackets: consider the Reeb fielddefined as

$$\mathbf{i}_X \alpha = 1 \qquad \text{and} \qquad \mathbf{i}_X d\alpha = 0$$

so that $X = \partial/\partial z$ and the foliation integrates the distribution $\mathcal{D}$ spanned by $\partial/\partial z$ and $\partial/\partial\vartheta$; notice that $\omega(X, \partial/\partial\vartheta) = 1$, and that its pull-backs to leaves are symplectic. In this way, the foliated complex structure defined by

$$J\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial \vartheta} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial \vartheta}\right) = -\frac{\partial}{\partial z}$$

is Kähler.

Example 3.2 *Consider S^3 as embedded in $\mathbb{R}^4$ (with global coordinates (x_1, x_2, x_3, x_4)) and equipped with the contact form induced by*

$$\alpha = x_1 dx_1 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3$$

Moreover take S^1 *as embedded in* $\mathbb{R}^2$ *(with global coordinates* (x_5, x_6)*): its tangent vector is given by*

$$V = -x_6 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_6}$$

and equip the product $M = S^3 \times S^1 \subset \mathbb{R}^6$ *with the 2-form*

$$\omega = d\alpha + \alpha \wedge d\vartheta$$

being ϑ *the 1-form dual to* V *defined as*

$$\vartheta = -x_6 dx_5 + x_5 dx_6$$

A global basis of the tangent bundle of S^3 *(as a sub-manifold of* $\mathbb{R}^4$*) is*

$$\begin{cases} X = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \\ Y = -x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} \\ Z = -x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} \end{cases}$$

Notice that X *is just the Reeb vector field, since*

$$\mathbf{i}_X \alpha = 1 \qquad \text{and} \qquad \mathbf{i}_X d\alpha = 0$$

so that the distribution $\mathcal{D}$ *spanned by* X *and* V *is integrable: hence we have Dirac brackets on* $M = S^3 \times S^1$*. Now, let us define an almost complex structure* J *on* M *as*

$$\begin{cases} J(X) = V \\ J(Y) = Z \\ J(Z) = -Y \\ J(V) = -X \end{cases}$$

As one may check, this is a complex structure and $\mathcal{D}$ *is* J*-invariant, so that we have defined a Poisson–Kähler structure on* $M = S^3 \times S^1$*.*

Remark 3.3 *The foliated complex structure* J *of the previous example is actually the restriction of the following integrable complex structure defined* *on an open dense set in* $\mathbb{R}^6$*:*

$$\begin{pmatrix} 0 & -\frac{x_1^2+x_4^2}{r_1^2} & \frac{x_1x_2+x_3x_4}{r_1^2} & \frac{x_2x_4-x_1x_3}{r_1^2} & \frac{x_3x_5+x_2x_6}{r_2^2} & \frac{x_2x_5-x_3x_6}{r_2^2} \\ \frac{x_1^2+x_4^2}{r_1^2} & 0 & \frac{x_1x_3-x_2x_4}{r_1^2} & \frac{x_1x_2+x_3x_4}{r_1^2} & \frac{x_3x_6-x_2x_5}{r_2^2} & \frac{x_3x_5+x_2x_6}{r_2^2} \\ -\frac{x_1x_2+x_3x_4}{r_1^2} & \frac{x_2x_4-x_1x_3}{r_1^2} & 0 & -\frac{x_2^2+x_3^2}{r_1^2} & \frac{x_4x_6-x_1x_5}{r_2^2} & \frac{x_4x_5+x_1x_6}{r_2^2} \\ \frac{x_1x_3-x_2x_4}{r_1^2} & -\frac{x_1x_2+x_3x_4}{r_1^2} & \frac{x_2^2+x_3^2}{r_1^2} & 0 & -\frac{x_4x_5+x_1x_6}{r_2^2} & \frac{x_4x_6-x_1x_5}{r_2^2} \\ -\frac{x_3x_5+x_2x_6}{r_1^2} & \frac{x_2x_5-x_3x_6}{r_1^2} & \frac{x_1x_5-x_4x_6}{r_1^2} & \frac{x_4x_5+x_1x_6}{r_1^2} & 0 & 0 \\ \frac{x_3x_6-x_2x_5}{r_1^2} & -\frac{x_3x_5+x_2x_6}{r_1^2} & -\frac{x_4x_5+x_1x_6}{r_1^2} & \frac{x_1x_5-x_4x_6}{r_1^2} & 0 & 0 \end{pmatrix}$$

where $r_1^2 = x_1^2 + \cdots + x_4^2$ *and* $r_2^2 = x_5^2 + x_6^2$*.*

A similar example may also be constructed on $S^{2n+1} \times S^1$, and, more generally, on each product $M \times S^1$ being M a compact contact manifold: we simply pick the Reeb field X on M and the tangent field V to the second factor S^1, thus we consider the Poisson structure given by Dirac brackets, and the foliated complex structure

$$J(X) = V \qquad \text{and} \qquad J(V) = -X$$

It's interesting that these manifolds are not Kähler (they have wrong Betti numbers...) but are Poisson with Kähler leaves. A more complex example is the following:

Example 3.4 *Let us consider the* Iwasawa manifold $I(3)$ *which, we remind, is the quotient* H/Γ *being*

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{z_1, z_1, z_3 \in \mathbb{C}} \cong \mathbb{C}^3$$

and Γ *the subgroup*

$$\Gamma = \left\{ \begin{pmatrix} 1 & m_1 & m_3 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{m_1, m_1, m_3 \in \mathbb{R}^3[\sqrt{-1}]}$$

acting on H *by multiplication.*

It is well-known that this manifold cannot be Kähler w.r.t. any other complex structure on it ([3]). Now we want to construct a Poisson structure on $I(3)$ of rank four with Kähler leaves and, to do this, we define Dirac brackets, and a global integrable complex structure, thus we need: 1) a non degenerate 2-form ω on $I(3)$; 2) a foliation of rank 4 whose leaves are symplectic w.r.t. pull-backs of ω; 3) a complex structure on $I(3)$ such that its restriction to any leave of the foliation gives rise to a Kähler structure.

Let us work on the group G with complex coordinates

$$(z_1, z_2, z_3) = (x_1 + \sqrt{-1}x_2, x_3 + \sqrt{-1}x_4, x_5 + \sqrt{-1}x_6)$$

Next we consider the following basis for $TI(3)$

$$\begin{aligned}
\xi_1 &= \frac{\partial}{\partial x_1} & \xi_4 &= \frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_5} + x_1\frac{\partial}{\partial x_6} \\
\xi_2 &= \frac{\partial}{\partial x_5} & \xi_5 &= \frac{\partial}{\partial x_6} \\
\xi_3 &= \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_5} + x_2\frac{\partial}{\partial x_6} & \xi_6 &= \frac{\partial}{\partial x_2}
\end{aligned}$$

and their dual 1-forms

$$\begin{aligned}
\alpha_1 &= dx_1 & \alpha_4 &= dx_4 \\
\alpha_2 &= dx_5 - x_1 dx_3 + x_2 dx_4 & \alpha_5 &= dx_6 - x_2 dx_3 - x_1 dx_4 \\
\alpha_3 &= dx_3 & \alpha_6 &= dx_2
\end{aligned}$$

Now define the non degenerate 2-form

$$\omega = \alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4$$

and take the distribution $\mathcal{D}$ spanned by $\{\xi_2, \xi_3, \xi_4, \xi_5\}$: since commutation rules for ξ_is are

$$[\xi_1, \xi_4] = \xi_5 \qquad [\xi_1, \xi_3] = \xi_2 \qquad [\xi_3, \xi_6] = -\xi_5 \qquad [\xi_4, \xi_6] = \xi_2$$

(remaning commutators being zero), we get an involutive distribution: moreover

$$d\omega(\xi_i, \xi_j, \xi_k) = 0$$

for any $i, j, k \in \{2, 3, 4, 5\}$, thus the pull-backs of ω to the leaves of $\mathcal{D}$ are closed and, since these pull-backs are non degenerate, we get a symplectic foliation, thus a regular Poisson structure. Finally consider the almost complex structure defined as

$$\begin{aligned}
J(\xi_1) &= -\xi_6 & J(\xi_2) &= \xi_5 & J(\xi_3) &= \xi_4 \\
J(\xi_4) &= -\xi_3 & J(\xi_5) &= -\xi_2 & J(\xi_6) &= \xi_1
\end{aligned}$$

Theorem 3.5 *J is an integrable complex structure on $I(3)$, which induces a foliated complex structure on the foliation which integrates $\mathcal{D}$, and Kähler structures on leaves w.r.t. the symplectic structures induced by ω.*

Remark 3.6 *Notice that the integrable complex structure J on $I(3)$ is calibrated by the 2-form ω, thus:*

$$\omega(JX, JY) = \omega(X, Y) \qquad \text{and} \qquad \omega(JX, X) > 0$$

This means that there exists a global hermitian metric $h(X, Y) = \omega(JX, Y)$ (which, of course, cannot be Kählerian).

Remark 3.7 *All leaves are compact, namely they are four dimensional tori, and we can identify them to the fibers of the fibration $p : I(3) \longrightarrow \mathbb{T}^2$ whose fiber over $[z_1]$ is given by*

$$p^{-1}([z_1]) = \left[\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

4 Poisson–Kähler structures

Since a non degenerate two-form does exist on a manifold if and only if there exists also an almost complex structure (and vice versa) it is natural to ask about a relationship between two such structures on a same manifold: in the case of Dirac brackets the foliation must be involved too, so we ask about foliated almost complex structures, thus endomorphisms J of the bundle $T\mathcal{F}$ of vectors tangent to leaves such that $J^2 = -Id$.

Definition 4.1 *Let $(M, \mathcal{F})$ be a foliated manifold: then a non degenerate foliated two-form ω and a foliated almost complex structure J are said to be* compatible *(and ω is said to be* calibrated *on J) if the following hold true:*

(1) $\omega(JX, JY)(x) = \omega(X, Y)(x)$

(2) $\omega(JX, X)(x) > 0$

(3) $i^*(d\omega)(x) = 0$

where i denotes the embedding of the leaf through any $x \in M$.

An *(almost) Poisson–Kähler manifold* is a Poisson manifold equipped with a foliated (almost) complex structure compatible with the foliated 2-form induced by the Poisson structure: of course symplectic leaves of a Poisson–Kähler manifolds are Kähler.

Proposition 4.2 *If $(M, \mathcal{F}, \omega)$ is a Dirac manifold then there exists a calibrated foliated almost complex structure on M.*

PROOF: It sufficies to consider the local case: take $M = \mathbb{R}^{2n} \times \mathbb{R}^{2m}$ with the foliation given by the product and

$$\omega = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{pmatrix}$$

Since the space of ω-calibrated foliated complex structures on $\mathbb{R}^{2(n+m)}$ is

$$\frac{Sp(2n) \times Sp(2m)}{U(n) \times U(m)} \tag{$*$}$$

the set of ω-calibrated foliated almost complex structures on a Dirac manifold is in one to one correspondence with the sections of the fiber bundle $E \to M$ with standard fiber given by (*), and since this homogeneous space is contractible, E always admits sections.

QED

Remark 4.3 *Notice that a $Sp(2n) \times Sp(2m)$-structure on a manifold is equivalent to give a pair of complementary almost symplectic distributions $(\mathcal{F}, \mathcal{G})$ (cf. [5]), so that any Dirac manifold has a complementary almost symplectic distribution (not necessarily integrable).*

Now we want to face the same questions in the general case of a regular Poisson structure: first of all notice that we can identify the set $\mathcal{C}$ of foliated complex structures on $\mathbb{R}^{2n+m}$ with the space $GL^+_{2n+m}(\mathbb{R})/GL_n(\mathbb{C}) \cdot GL_m(\mathbb{R})$. Consider the Poisson structure on $\mathbb{R}^{2n} \times \mathbb{R}^m$ defined as[5]

$$\pi_0 = \begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix}$$

[5]By ω_0 we mean the standard symplectic structure in $\mathbb{R}^{2n}$.

We are interested in the subspace

$$\mathcal{C}(\pi_0) = \mathcal{C} \cap Ps(2n, m)$$

where $Ps(2n, m)$ is the Poisson group of invertible linear maps $A : \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ such that $A^T \pi_0 A = \pi_0$. Thus we are considering the space $\mathcal{C}(\pi_0)$ of foliated complex structures which leave π_0 invariant.

Of course, both $Sp(2n)$ and $GL_m(\mathbb{R})$ are subgroups of $Ps(2n, m)$: moreover we have the following simple

Lemma 4.4 $Ps(2n, m) = Sp(2n) \cdot GL_m(\mathbb{R}) \cdot \hom(\mathbb{R}^{2n}, \mathbb{R}^m)$

Now, an element of $\mathcal{C}(\pi_0)$ is the image of a Poisson map: indeed the Poisson group $Ps(2n, m)$ acts on this set by the usual conjugation action:

$$A \cdot J := AJA^{-1}$$

This action is transitive: indeed, if we denote by $J_0 \in \mathcal{C}(\pi_0)$ the matrix

$$J_0 = \begin{pmatrix} \omega_0 & 0 \\ 0 & I \end{pmatrix}$$

and let W be an arbitrary subspace of dimension $2n$ endowed with a complex structure induced by a foliated complex structure $J \in \mathcal{C}(\pi_0)$, after picking a Darboux basis on W $(v_1, ..., v_n, w_1, ..., w_n)$ such that $Jv_i = w_i$ ([6]) we see that there exists a unitary transformation $U : \mathbb{R}^{2n} \to W$ which extends to a transformation $T : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^{2n} \times \mathbb{R}^m$ in $U(n) \times GL_m(\mathbb{R})$. Moreover, the isotropy group at the point W_0 is $U(n) \times GL_m(\mathbb{R})$, hence we have proved the following

Proposition 4.5 $\mathcal{C}(\pi_0) = Ps(2n, m)/U(n) \times GL(m, \mathbb{R})$

Now, by the previous lemma, $\mathcal{C}(\pi_0)$ becomes homeomorphic to

$$\frac{Sp(2n)}{U(n)} \times \mathbb{R}^{2nm}$$

and hence contractible, so that, given a regular Poisson structure (M, π) on a manifold, the set $\mathcal{C}(\pi)$ of foliated complex structures (defined on the symplectic distribution induced by π) compatible with π is not empty:

Corollary 4.6 *Let (M, π) be a regular Poisson manifold and $H(M) \subset TM$ its distribution, then there exist a foliated almost complex structure J and a J-Hermitian foliated metric h which is almost Kähler w.r.t. π.*

We have seen that a regular Poisson structure admits always almost Poisson–Kähler structures: on the other hand we want now to stress that, given an almost complex structure tangent to a foliation, it is not always possible to find out a Poisson structure such that the pair (π, J) is almost Poisson–Kähler.

Let J be a foliated almost complex structure on a foliated manifold $(M, \mathcal{F})$: since we will deal with jets computations, we can still assume $M = \mathbb{R}^{2n+m}$ (where $2n$ is the dimension of leaves, which is even since there is a foliated almost complex structure J on them) and we will also assume that leaves are given by planes

$$x_{2n+1} = c_1 \, , \quad \dots \quad , \, x_{2n+m} = c_m$$

Let π be a Poisson structure compatible with J such that $\pi(0) = \pi_0$ the latter being the canonical Poisson structure of rank $2n$ in $\mathbb{R}^{2n+m}$.

From Weinstein's splitting theorem it follows that there exists a (local) diffeomorphism $\varphi : \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ such that

$$\varphi_*(\pi) = \pi_0$$

According to this diffeomorphism, the structure J will change into $\tilde{J} = \varphi_*(J)$. Without loss of generality, we can assume that φ_* evalued at 0 is the identity and that

$$J(0) = J_0$$

Then J is π-compatible if and only if $\tilde{J}$ is π_0-compatible, which can be expressed by the following conditions

$$\begin{cases} \omega_0(\tilde{J}X, \tilde{J}Y) = \omega_0(X, Y) \\ \omega_0(\tilde{J}X, X) > 0 \end{cases} \tag{1}$$

for all X and Y vector fields tangent to leaves (being $X \neq 0$ in the second equation). Let us denote φ_* in matrix form as

$$\varphi_*(x) = \begin{pmatrix} \varphi_{11}(x) & \varphi_{12}(x) \\ 0 & \varphi_{22}(x) \end{pmatrix}$$

and $J(x)$ as

$$J(x) = \begin{pmatrix} J_{11}(x) & J_{12}(x) \\ 0 & J_{22}(x) \end{pmatrix}$$

being $J_{11}(0) = J_0$.

Theorem 4.7 *If there exists a Poisson structure π on M with Hamiltonian foliation given by $\mathcal{F}$ which is almost Poisson–Kähler (w.r.t. J) then the following hold true*

$$\left[\left.\frac{\partial\varphi_{11}}{\partial x_i}\right|_0 - \left.\frac{\partial\varphi_{11}^T}{\partial x_i}\right|_0, J(0)\right] = \left.\frac{\partial J_{11}^T}{\partial x_i}\right|_0 - \left.\frac{\partial J_{11}}{\partial x_i}\right|_0$$

$(i = 1, ..., 2n+m)$.

PROOF: Let us consider the first equation of (1):

$$\omega_0((\varphi_* J\varphi_*^{-1})(X), (\varphi_* J\varphi_*^{-1})(Y)) = \omega_0(X, Y)$$

Now looking at the first order jets in both hands of this equation, we get that (being $(e_1, ..., e_{2n})$ the canonical symplectic basis on the leaves)

$$j_0^1(\omega_0((\varphi_* J\varphi_*^{-1})(e_r), (\varphi_* J\varphi_*^{-1})(e_s))) = j_0^1(\omega_0(e_r, e_s))$$

if and only if

$$\left(\left.\left(\left[\frac{\partial\varphi_*}{\partial x_i}, J\right] + \frac{\partial J}{\partial x_i}\right)\right|_0 e_r\right)^T \pi_0 J(0)e_s + (J(0)e_r)^T \pi_0 \left.\left(\left[\frac{\partial\varphi_*}{\partial x_i}, J\right] + \frac{\partial J}{\partial x_i}\right)\right|_0 e_s = 0$$

which is equivalent to say that the following matrix

$$\left.\left[\frac{\partial\varphi_{11}}{\partial x_i}, J_0\right]\right|_0 + \left.\frac{\partial J}{\partial x_i}\right|_0$$

is symmetric.

QED

Needless to say, a similar result holds for Dirac manifolds.

Example 4.8 *A foliated almost complex structures which admits no compatible Poisson structures: take $\mathbb{R}^7$ with coordinates $(x_1, ..., x_7)$ endowed with the trivial foliation of codimension one, just given by*

$$x_7 = const.$$

Consider the foliated almost complex structure whose matrix, in the given coordinates, is $\begin{pmatrix} J(x) & H(x) \\ 0 & \psi(x) \end{pmatrix}$, being $J(x)^2 = -Id$ and $\psi(x)$ a non vanishing *smooth function. The compatibility condition of Theorem 4.7 becomes*

$$\begin{cases} \left(\frac{\partial J_{62}}{\partial x_1} - \frac{\partial J_{26}}{\partial x_1}\right)\Big|_0 + \left(\frac{\partial J_{16}}{\partial x_2} - \frac{\partial J_{61}}{\partial x_2}\right)\Big|_0 + \left(\frac{\partial J_{51}}{\partial x_3} - \frac{\partial J_{15}}{\partial x_3}\right)\Big|_0 + \\ \qquad + \left(\frac{\partial J_{32}}{\partial x_4} - \frac{\partial J_{23}}{\partial x_4}\right)\Big|_0 + \left(\frac{\partial J_{13}}{\partial x_5} - \frac{\partial J_{31}}{\partial x_5}\right)\Big|_0 + \left(\frac{\partial J_{21}}{\partial x_6} - \frac{\partial J_{12}}{\partial x_6}\right)\Big|_0 = 0 \\ \left(\frac{\partial J_{23}}{\partial x_1} - \frac{\partial J_{32}}{\partial x_1}\right)\Big|_0 + \left(\frac{\partial J_{31}}{\partial x_2} - \frac{\partial J_{13}}{\partial x_2}\right)\Big|_0 + \left(\frac{\partial J_{12}}{\partial x_3} - \frac{\partial J_{21}}{\partial x_3}\right)\Big|_0 + \\ \qquad + \left(\frac{\partial J_{62}}{\partial x_4} - \frac{\partial J_{26}}{\partial x_4}\right)\Big|_0 + \left(\frac{\partial J_{16}}{\partial x_5} - \frac{\partial J_{61}}{\partial x_5}\right)\Big|_0 + \left(\frac{\partial J_{51}}{\partial x_6} - \frac{\partial J_{15}}{\partial x_6}\right)\Big|_0 = 0 \end{cases} \tag{2}$$

The following foliated almost complex structure

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & \psi_1 \\ 0 & 0 & 0 & 0 & -1 & 0 & \psi_2 \\ \varphi(x) & 0 & 0 & 0 & 0 & -1 & \psi_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & \psi_4 \\ 0 & 1 & 0 & 0 & 0 & 0 & \psi_5 \\ 0 & 0 & 1 & -\varphi(x) & 0 & 0 & \psi_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_7 \end{pmatrix}$$

where $\varphi, \psi_1, ..., \psi_7$ are smooth functions on $\mathbb{R}^7$ such that:

$$\frac{\partial \varphi}{\partial x_2}(0) \neq 0 \qquad \text{and} \qquad \psi_7(x) \neq 0$$

does not satisfy the second equation of the system (2), and so cannot be almost Poisson–Kähler. By taking $\mathbb{R}$-periodic functions this almost complex structure passes on to the torus $\mathbb{T}^7$.

We observe that the latter example can be extended to any foliation of rank greater than four. On the other hand, when the rank is four, the compatibility conditions of (2) are always satisfied.

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