The algebra of Poisson brackets

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1 Introduction

Poisson brackets were introduced by Joseph-Louis Lagrange and his student Simon-Denis de Poisson at the beginning of XIX century, as an algorithm useful to produce solutions of the equations of motion: of course no one ignores the definition of Poisson brackets given by Poisson himself

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$$

where $f, g \colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are smooth functions and (q, p) Lagrange's canonical coordinates. This produces a new smooth function which has the following remarkable property stressed by Poisson: *whenever I and J are constant of motion for a Hamiltonian system also $\{I, J\}$ is.*

In the thirties of the XIX century, Jacobi discovered a very simple proof of Poisson result: he remarked that if f is a function, the map $g \mapsto \{f, g\}$ is a vector field, because of the Leibniz identity for Poisson brackets, which in turn rests upon Leibniz rule for the differential of a product of functions:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$

We call such a vector field the *Hamiltonian vector field* associated to the function f, and denote it by X_f. Now Jacobi wondered about the vector field $X_{\{f,g\}}$: can it be expressed in a simple way in terms of f and g? The answer given by Jacobi is

$$X_{\{f,g\}} = [X_f, X_g]$$

where square brackets denote the commutator of vector fields.

Now come back to Poisson theorem: if I and J are constant along the trajectories of motion of a Hamiltonian system, and if H is the Hamiltonian function, we have $X_I(H) = X_J(H) = 0$, so that, by Jacobi identity

$$X_{\{I,J\}}(H) = [X_I, X_J](H) = X_I(\{J, H\}) - X_J(\{I, H\}) = 0$$

hence $\{I, J\}$ is a constant of the motion too.

What Jacobi discovered was in fact the first example of a Lie algebra, since, by applying the vector field $X_{\{f,g\}}$ to a function h, we find that

$$\{\{f, g\}, h\} = \{\{f, g\}, h\} - \{\{g, f\}, h\}$$

which is the nowadays familiar Jacobi identity.

In the seventies of XIX century Marius Sophus Lie began his deep researches on the geometry of P.D.E. which had their achievement in the monumental trilogy *Theorie der Transformationsgruppen*; in these hundreds of pages, among myriads of other things, a more systematic study of Poisson brackets is started, and Lie describes new examples of Poisson brackets, whose nature is different from the Poisson and Lagrange's ones[1].

The unifying model for both Poisson and Lie brackets is the definition of Poisson algebra, which can be stated at different levels of generality: the aim of this note is precisely to sketch a general theory of Poisson brackets.

2 Definitions and examples

A purely algebraic motivation for the introduction of Poisson brackets is the following: they combine both associative and Lie structures.

Definition 2.1 *A* Poisson algebra *over a (commutative) ring (with unit)* $\mathbb{K}$ *is a triple* $(A, \cdot, \{\,\})$ *where* $(A, \cdot)$ *is an associative* $\mathbb{K}$*-algebra and* $(A, \{\,\})$ *is a Lie* $\mathbb{K}$*-algebra, such that the following identity*

$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$$

is satisfied for each $a, b, c \in A$.

[1] These brackets are now called *Lie–Poisson brackets* and were ignored by mathematicians, until Kirillov, Konstant and Souriau redefined them in the context of the theory of Lie groups representations and geometric quantization theory.

So the axioms for a Poisson algebra are the following:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$\{a, b\} + \{b, a\} = 0$$
$$\{\{a, b\}, c\} + \{\{c, a\}, b\} + \{\{b, c\}, a\} = 0$$
$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$$

We well deal only with algebras on a field $\mathbb{K}$, and moreover we will be only interested in *commutative* Poisson algebra, thus in algebras such that

$$a \cdot b = b \cdot a$$

for each $a, b \in A$; the main reason is that Poisson algebras arise mainly as algebras of functions (with the associative structure given by the point wise multiplication).

We start with some trivial examples:

Example 2.2 *Every Lie algebra is a Poisson algebra w.r.t. the null associative product: $a \cdot b = 0$, and every associative algebra is a Poisson algebra w.r.t. the null Poisson bracket: $\{a, b\} = 0$; such an algebra is called* null Poisson algebra.

Example 2.3 *An associative algebra A is a Poisson algebra if we put $\{a, b\} = ab - ba$; indeed*

$$(*) \quad \{ab, c\} = (ab)c - c(ab) = a(bc) - a(cb) + (ac)b - (ca)b = a\{b, c\} + \{a, c\}b$$

so we get a Poisson algebra; however this Poisson structure is completely determined by the associative one. Vice versa, if $\mathfrak{g}$ is a Lie algebra then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Poisson algebra by means of the same computation performed above.

Example 2.4 *Consider a vector space V on $\mathbb{K}$, its dual space V^* and the commutative algebra A generated by the linear functions on $V \oplus V^*$ (we assume for simplicity the reflexivity of V although it is not needed); we can look at A as the symmetric algebra Sym $(V^* \oplus V)$ (if V is a topological vector space we consider continuous symmetric tensors). Now, if $\varphi \oplus v, \psi \oplus w \in V^* \oplus V$, define*

$$\{\varphi \oplus v, \psi \oplus w\} = \varphi(w) - \psi(v)$$

(notice that $\{V^*, V\} = 0$). This gives a $\mathbb{K}$-bilinear skew-symmetric map on $V^* \oplus V$, that we extend to A by requiring

$$\{a+b, c\} = \{a, c\} + \{b, c\}, \qquad \{ab, c\} = a\{b, c\} + b\{a, c\}, \qquad \{\mathbb{K}, A\} = 0$$

Since A is generated (as an algebra) by $V^* \oplus V$, this defines a bilinear skew-symmetric operation which, by definition, satisfies Leibniz identity, while Jacobi identity may be verified by a simple induction (viewing A as the symmetric algebra the induction follows the grading of the algebra). This Poisson algebra is called symplectic algebra *on V.*

Notice that this Poisson algebra has a center which at least contains $\mathbb{K}$: but in fact $\mathbb{K}$ is precisely the center, since the skew-symmetric scalar product $\{\,\}\colon V^* \oplus V \times V^* \oplus V \longrightarrow \mathbb{K}$ is non degenerate[2]. The space $V^* \oplus V$ becomes in this way a symplectic vector space, with symplectic form given by $\{\,\}$ (and of course every symplectic space can be so obtained, by Darboux theorem).

For example if $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^n$ then this construction gives the usual symplectic brackets on $\mathbb{R}^{2n}$ viewed as $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$; we recover canonical Poisson brackets if we consider a basis $(e^1, ..., e^n)$ of V, its dual basis $(\varepsilon_1, ..., \varepsilon_n)$ in V^*, and notice that (we use Einstein convention on indexes)

$$\{\varphi \oplus v, \psi \oplus w\} = \{\alpha^i \varepsilon_i + a_i e^i, \beta^j \varepsilon_j + b_j e^j\} = \alpha^i b_i - a_i \beta^i$$

Now, the symmetric algebra on $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$ can be seen as the polynomial algebra $\mathbb{R}[q_1, ..., q_n, p_1, ..., p_n]$; by our definition of Poisson brackets:

$$\{q_i, p_j\} = \delta_{ij} \qquad \text{and} \qquad \{q_i, q_j\} = \{p_i, p_j\} = 0$$

for each $i, j = 1, ..., n$. Because we require both Leibniz and bilinear identity we get for two general polynomials $A(q,p) = \sum_{\alpha,\beta} a_{\alpha\beta} q^\alpha p^\beta$ and $B(q,p) = \sum_{\eta,\kappa} b_{\eta\kappa} q^\eta p^\kappa$ (for the sake of simplicity we work out the computation for $n = 1$, so that the multi-index α is a single index i, and so on) that[3]:

$$\begin{aligned}\{A, B\}(q,p) &= a_{ij} b_{hk} \{q^i p^j, q^h p^k\} \\ &= a_{ij} b_{hk} \left(ik q^{i+h-1} p^{j+k-1} \{q, p\} + jh q^{i+h-1} p^{j+k-1} \{p, q\} \right) \\ &= a_{ij} b_{hk} \left(\frac{\partial q^i p^j}{\partial q} \frac{\partial q^h p^k}{\partial p} - \frac{\partial q^i p^j}{\partial p} \frac{\partial q^h p^k}{\partial q} \right) = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\end{aligned}$$

[2] Because if $\{\varphi \oplus v, \psi \oplus w\} = 0$ for each $w \in V$, $\psi \in V^*$, then, for w and ψ such that $\varphi(w) = 1$ and $\psi(v) = 0$ (they exists for trivial reasons if $\dim V < \infty$ and by some version of Hahn–Banach theorem if $\dim V = \infty$): $0 = \{\varphi \oplus v, \psi \oplus w\} = 1$ which is absurd.

[3] We use here the following consequence of Leibniz formula for commutative algebras:

$$\begin{aligned}\{a^n b^m, c^h d^k\} = {} & nha^{n-1} b^m c^{h-1} d^k \{a, c\} + nka^{n-1} b^m c^h d^{k-1} \{a, d\} \\ & + mha^n b^{m-1} c^{h-1} d^k \{b, c\} + mka^n b^{m-1} c^h d^{k-1} \{b, d\}\end{aligned}$$

Of course in symplectic geometry one considers smooth functions rather than polynomial ones, but Poisson brackets on polynomials determine Poisson brackets on smooth functions, for example by density of topological vector spaces, and in fact our example has a geometric counterpart in the notion of symplectic manifold: if (S, ω) is a symplectic manifold then the algebra $C^\infty(S)$ is a Poisson algebra with Poisson brackets given by

$$\{f, g\} = \omega(X_f, X_g)$$

where X_f, X_g are the Hamiltonian vector fields generated by f and g; locally these brackets are determined by the same commutation rules $\{q_i, p_j\} = \delta_{ij}$ and $\{q_i, q_j\} = \{p_i, p_j\} = 0$ we stated above.

Example 2.5 *Be V a vector space such that the (topological, if it is the case) dual V^* is a Lie algebra $\mathfrak{g}$ w.r.t. some fixed Lie brackets $[\,]$; since $\mathfrak{g}$ is the space of linear functions on V, the commutative algebra generated by linear functions on V is the symmetric algebra on $\mathfrak{g}$: but Poincaré–Birkhoff–Witt theorem affirms that there exists an isomorphism*

$$\mathbf{Gr}\,\mathcal{U}(\mathfrak{g}) \cong \mathit{Sym}\,(\mathfrak{g})$$

between the graded algebra associated to the filtration of the universal enveloping algebra of $\mathfrak{g}$ and the symmetric algebra over $\mathfrak{g}$, so that we can put on $\mathbf{Gr}\,\mathcal{U}(\mathfrak{g})$ (thus on $\mathit{Sym}\,(\mathfrak{g})$) a Poisson algebra structure as follows: remember that $\mathfrak{g}$ injects into $\mathcal{U}(\mathfrak{g})$ and that $\mathcal{U}(\mathfrak{g})$ is filtered as (cfr. [8, §3])

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots \subset \mathcal{U}_k \subset \dots$$

where $\mathcal{U}_0 = \mathbb{K}$ and $\mathcal{U}_k$ are generated by $\mathbb{K}$ and by products $x_1 \dots x_h$ (with $h \leq k$) of elements of the Lie algebra $\mathfrak{g}$ (so that $\mathcal{U}_1 = \mathbb{K} \oplus \mathfrak{g}$). Hence

$$\mathbf{Gr}\,\mathcal{U}(\mathfrak{g}) = \bigoplus_{k \geq 0} \frac{\mathcal{U}_{k+1}}{\mathcal{U}_k}$$

So an element of degree k is an equivalence class $[x]$ of products of at most k elements of $\mathfrak{g}$; then we can define a skew-symmetric bilinear map $\{\,\} : \mathcal{U}_k \times \mathcal{U}_h \longrightarrow \mathcal{U}_{h+k-1}$ as

$$\{[x], [y]\} := [xy - yx]$$

(notice that the image is in $\mathcal{U}_{h+k-1}$ and not in $\mathcal{U}_{h+k}$ by the universal property of the enveloping algebra: for example, if x, y have degree one, and so belong to $\mathfrak{g}$, $xy - yx$ has degree one too, being precisely $[x, y]$).

In this way the algebra $\mathbf{Gr}\,\mathcal{U}(\mathfrak{g})$ becomes a Poisson algebra w.r.t. the brackets $\{\,\}$: Leibniz identity is easily verified, while Jacobi identity follows from the usual Jacobi identity in the Lie algebra $\mathfrak{g}$ by an easy induction on the degree: this algebra is called *Lie–Poisson* algebra over $\mathfrak{g}$.

The space $\mathrm{Sym}\,(\mathfrak{g})$ can be viewed as the algebra of polynomials on the space $\mathfrak{g}^*$, and in this case this Poisson structure was discovered by Lie who gave explicit formulas and used it to prove his inverse third theorem. We want to recover here his formula: be $V = \mathbb{R}^n$, so that $V^* = \mathfrak{g} = (\mathbb{R}^n)^*$ and fix a basis $(e^1, ..., e^n)$ in V and a basis $(\varepsilon_1, ..., \varepsilon_n)$ in $\mathfrak{g}$. Then Lie brackets are determined by their structure constants

$$[\varepsilon_i, \varepsilon_j] = c_{ij}^k \varepsilon_k$$

Now consider two polynomials $f, g\colon V \longrightarrow \mathbb{R}$ (in fact the argument only require $f, g \in C^\infty(\mathbb{R}^n)$) and define

$$\{f, g\}(v) = c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

$(v = x_i e^i)$ which is precisely as Lie defined Lie–Poisson brackets.

More generally we could consider an associative $\mathbb{K}$-algebra A with a unit 1 and a filtration, thus such that $\bigcup_{n\in\mathbb{N}} A_n$ for a sequence of subspaces

$$A_0 \subset A_1 \subset A_2 \subset ... \subset A_{n-1} \subset A_n \subset A_{n+1} \subset ...$$

in such a way that the product is compatible with the filtration:

$$A_i \cdot A_j \subset A_{i+j}$$

Now suppose that A satisfies to the following condition (stressed by Krasilščik and Vinogradov [5])

$$(KV) \qquad [A_i, A_j] \subset A_{i+j-1}$$

where $[\,]$ is the commutator induced by the associative product (we are in other words assuming that $ab - ba \in A_{i+j-1}$ if $a \in A_i$ and $b \in A_j$). Therefore the graded algebra

$$\mathbf{Gr}\,A = \bigoplus_{n\geq 1} \frac{A_n}{A_{n-1}}$$

is a Poisson algebra w.r.t. the associative product and the commutator passed onto the quotient: this can be seen exactly as in the case of $A = \mathcal{U}(\mathfrak{g})$.

This generalization of the Lie construction is only seeming: indeed a filtered algebra which satisfies (KV) is always the enveloping algebra of a certain

Lie algebra; more precisely, it suffices to consider A_1 and notice that condition (KV) becomes $[A_1, A_1] \subset A_1$, so that A_1 is a Lie algebra whose enveloping algebra is, by definition, A itself.

As an example we take the algebra of differential operators, which we can define in a purely algebraic manner as follows (cfr.[5], [3]): consider an associative algebra A and define, for a fixed $a \in A$, the map

$$\mu_a \colon A \longrightarrow A$$

of left multiplication: $\mu_a(b) = ab$, and, if $X \in \mathrm{End}_{\mathbb{K}}(A)$, put

$$D_a(X) = [\mu_a, X]$$

This is a $\mathbb{K}$-linear map both in a and in X; moreover define

$$D_{a_0 a_1 \ldots a_k} = D_{a_0} D_{a_1} \ldots D_{a_k}$$

Definition 2.6 *A* differential operator *is an $\mathbb{K}$-linear operator $X \colon A \longrightarrow A$ such that*

$$\forall a_0\ \forall a_1 \ldots\ \forall a_n \quad D_{a_0 a_1 \ldots a_k}(X) = 0$$

n is said to be the order *of the operator.*

For instance when $A = C^\infty(M)$ (algebra of smooth function on a manifold) we recover the usual concept of a differential operator: the condition that D is a differential operator of order n can be written simply as $D^{n+1}(X) = 0$. Consider now

$$\mathfrak{D}_n(A) = \{X \in \mathrm{End}_{\mathbb{K}}(A) \,|\, D^{n+1}(X) = 0\}$$

Evidently $\mathfrak{D}_0(A) = \{\mu_a\}_{a \in A}$, and $\mathfrak{D}_n(A) \subset \mathfrak{D}_{n+1}(A)$ so that the set

$$\mathfrak{D}(A) = \bigcup_{n \in \mathbb{N}} \mathfrak{D}_n(A)$$

is an associative filtered algebra: one can prove, by induction on the order of operators (cfr. [3]), that

$$[\mathfrak{D}_n(A), \mathfrak{D}_m(A)] \subset \mathfrak{D}_{n+m-1}(A)$$

so that we can define on $\mathbf{Gr}\,\mathfrak{D}(A)$ a Poisson structure: this graded algebra is nothing but the algebra of symbols of differential operators.

For example be V a vector space and $\mathfrak{D}(V)$ the algebra of differential operators obtained by considering the symmetric algebra $A = \mathrm{Sym}\,(V)$; it is, once a basis $(e^1, ..., e^n)$ is fixed in V, the algebra of polynomials, and a differential operators can be written as

$$X = \sum_{|\alpha| \leq n} p_\alpha \partial^\alpha$$

where $\alpha = (a_1, ..., a_n)$ is a multi-index and $\partial^\alpha = \partial_1^{a_1} ... \partial_n^{a_n}$, being ∂_i the derivation associated to the element e^i $(\partial_i e^j = \delta_{ij})$. The symbol of the operator X is the function $\sigma_X \colon V \times V^* \longrightarrow \mathbb{K}$ defined as

$$\sigma_X(v, \varphi) = \sum_{|\alpha|=n} p_\alpha(v) \varphi_1^{a_1} ... \varphi_n^{a_n}$$

where $\varphi = \sum_i \varphi^i \varepsilon_i$ in the dual basis $(\varepsilon_1, ..., \varepsilon_n)$ of $(e^1, ..., e^n)$. One can check that

$$\sigma_{XY} = \sigma_X \sigma_Y$$

and in fact a symbol belongs to $\mathfrak{D}(V)$. Since it is a polynomial function,

$$\sigma \colon \mathfrak{D}(V) \longrightarrow \mathrm{Sym}\,(V \times V^*)$$

passes onto the quotient and gives an isomorphism (cfr. [3]) of associative algebras

$$\overline{\sigma} \colon \mathbf{Gr}\,\mathfrak{D}(V) \longrightarrow \mathrm{Sym}\,(V \times V^*)$$

The former one is a Poisson algebra, since it is associated to a filtered one satisfying (KV) condition, and the latter is a symplectic Poisson algebra.

This is an abstract example of symplectic reduction, a fundamental phenomenon in symplectic geometry (cfr. [1]).

3 The category of Poisson algebras

Poisson algebras of course do form (the objects of) a category, whose morphisms are *Poisson maps*, thus morphisms $f \colon A \longrightarrow B$ of associative algebras which are also Lie algebra morphisms:

$$f\{a, b\} = \{f(a), f(b)\} \qquad \text{and} \qquad f(ab) = f(a)f(b)$$

Obviously a *Poisson subalgebra* of a Poisson algebra A is both an associative and a Lie subalgebra of A, and a Poisson subalgebra is a *Poisson ideal* if it is both an associative and a Lie ideal in A. The most important Poisson subalgebra of a given Poisson algebra A is its *Casimir subalgebra*, whose elements are called *Casimir elements*:

$$\mathrm{Cas}\, A = \{c \in A \,|\, \forall a \in A \quad \{a, c\} = 0\}$$

Notice that $\operatorname{Cas} A$ is a Lie but not a Poisson ideal (it is the center of the Lie algebra A).

Consider for instance the symplectic algebra A of a vector space V: its Casimir subalgebra is reduced to the algebra of constant symmetric functions $\mathbb{K}$.

Of course the usual algebraic concepts apply to the category of Poisson algebras: we can perform sum, direct sum, intersection and quotients in the Poisson category. More interestingly we can also define tensor products, as follows: if A_1 and A_2 are Poisson algebras then, on the space $A_1 \otimes A_2$ we put a Poisson algebra structure by defining the following operations:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (b_1 b_2)$$

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} = \{a_1, b_1\} \otimes a_2 b_2 + a_1 b_1 \otimes \{a_2, b_2\}$$

for $a_i, b_i \in A_i$. It's easy to check that with these two operations the space $A_1 \otimes A_2$ becomes a Poisson algebra, which of course is said to be the *tensor product* of A_1 and A_2. For example, when $A_1 = \operatorname{Sym}(V_1)$ and $A_2 = \operatorname{Sym}(V_2)$ we have $A_1 \otimes A_2 = \operatorname{Sym}(V_1 \oplus V_2)$.

If $(A, \cdot, \{\,\})$ is a Poisson algebra, for $a \in A$, if we define a $\mathbb{K}$-endomorphism $X_a \colon A \longrightarrow A$ as

$$X_a(b) = \{a, b\}$$

Leibniz identity says that X_a is a derivation of the associative algebra $(A, \cdot)$, and Jacobi identity says that X_a is a derivation of the Lie algebra $(A, \{\,\})$.

Definition 3.1 *A* Hamiltonian derivation *in A is a derivation of the form X_a for some $a \in A$. A* canonical *derivation is a derivation both of the associative and of the Lie algebra A.*

Of course canonical derivations form a Lie subalgebra $\operatorname{Can}(A)$ of $\operatorname{End}(A)$ (the Lie algebra of $\mathbb{K}$-linear operators $A \longrightarrow A$), and Hamiltonian derivations form a Lie ideal in $\operatorname{Ham}(A)$. Not every canonical derivation is a Hamiltonian one, and there exists the exact sequence of Lie algebras

$$0 \longrightarrow \operatorname{Cas} A \xrightarrow{i}\longrightarrow A \xrightarrow{X}\longrightarrow \operatorname{Ham} A \longrightarrow 0$$

where i is the injection. (This terminology is borrowed from Mechanics: if M is a symplectic manifold then $A = C^\infty(M)$ is a Poisson algebra, and Hamiltonian and canonical derivations correspond to Hamiltonian and locally Hamiltonian vector fields.)

The category of Poisson algebras has, of course, a “geometric” dual. Be A a Poisson algebra: then we can consider its spectrum, thus the set $\operatorname{Spec} A$

of maximal ideals; if A is commutative we can repeat the usual arguments of Algebraic Geometry and Functional Analysis to give to Spec A some topology. It suffices to consider elements of A as "points" $\chi \in \mathrm{Spec}\,(A)$ in the following familiar way:

$$a(\chi) = \chi(a)$$

(we identify maximal ideals and multiplicative functionals on the algebra). So we can consider the weak topology w.r.t. these functions on A.

Example 3.2 *If $A = C^\infty(M)$ where M is a smooth manifold then of course, as a set, Spec $(A) = M$. Moreover our topology coïncides in this case with the manifold topology since a set is closed if and only if it is the zero level set of a smooth function (Whithney's theorem).*

Example 3.3 *If $A = C(X)$ (complex continuous functions on a Hausdorff space) then Spec (A) is homeomorphic to X, as follows from Gel'fand–Naijmark theory.*

Now consider the algebra Cas A of Casimir elements of some Poisson algebra A, and its spectrum Spec Cas A with its topology. Obviously there exists a surjection

$$\Pi\colon \mathrm{Spec}\,A \longrightarrow \mathrm{Spec}\,\mathrm{Cas}\,A \longrightarrow 0$$

corresponding to the injection Cas $A \subset A$: thus, in some sense, the topological space Spec A defines a fibration on the space Spec Cas A.

Theorem 3.4 *Fibers of the map Π are spectra of symplectic Poisson algebras.*

PROOF: Take $\mathfrak{m} \in \mathrm{Spec}\,\mathrm{Cas}\,A$ and $\Pi^{-1}(\mathfrak{m})$: it is the set of maximal ideals which contain the ideal $\mathfrak{m}$. Now, for each $\mathfrak{M} \in \Pi^{-1}(\mathfrak{m})$, consider the quotient $A_{\mathfrak{M}} = \mathfrak{M}/\mathfrak{m}$: it is an associative algebra which is Poisson w.r.t the following brackets:

$$\{a + \mathfrak{m}, b + \mathfrak{m}\} = \{a, b\} + \mathfrak{m}$$

(where $a, b \in \mathfrak{M}$). This definition makes sense because $\mathfrak{m} \subset \mathrm{Cas}\,A$, and these brackets are really Poisson since $\{\,\}$ on A are; now compute Casimir elements for these brackets: if $c + \mathfrak{m}$ is such an element then, for each $a \in \mathfrak{M}$:

$$\{a + \mathfrak{m}, c + \mathfrak{m}\} = \{a, c\} + \mathfrak{m}$$

must belong to $\mathfrak{m}$, which means that $c+\mathfrak{m}$ defines an element in $\mathrm{Cas}\,A/\mathfrak{m} \cong \mathbb{K}$, therefore c is a constant. Hence brackets defined on $A_{\mathfrak{M}}$ are symplectic.

Q.E.D.

Notice that Π is continuous by definition.

4 Poisson calculus

Be A an associative and *commutative* $\mathbb{K}$-algebra with unit: since it is commutative, the correspondence $M \mapsto \operatorname{Der}(A, M)$ which assign to an A-module the module of derivations $D\colon A \longrightarrow M$, is a functor. *Kähler differentials* on A are by definition a representation of this functor, i.e. a pair (Ω_A, d) where Ω is an A-module and $d\colon A \longrightarrow \Omega_A$ a derivation, such that each derivation $\delta\colon A \longrightarrow M$ in an A-module M splits as $\delta = \mu \circ d$ where $\mu\colon \Omega_A \longrightarrow M$ is a morphism of A-modules (in other words, for each A-module M we have $\operatorname{Der}(A, M) = \hom_A(\Omega_A, M)$). Of course one must show that the module of differential exists: it can be constructed in various ways, cfr. [7, §1]; we look at it as the quotient of the A-module generated by the symbols $\{da\}_{a\in A}$ modulo the following relations

$$d(a+b) = da + db\,, \qquad d(ab) = adb + bda\,, \qquad d(1) = 0$$

We can perform differential calculus via Kähler differentials, in a purely algebraic way, and repeat all usual constructions of Cartan calculus: contractions, exterior derivative, Lie derivative; consider exterior powers $\Omega_A^k = \bigwedge^k \Omega_A$ of Ω_A, call their elements *differential forms* of degree k, and extend d to a graded differential $d\colon \Omega_A^k \longrightarrow \Omega_A^{k+1}$ on this graded algebra

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$

Now we define the *contraction* $\mathbf{i}\colon \operatorname{Der} A \times \Omega_A^k \longrightarrow A$ of a form on a derivation by extending the bilinear pairing

$$\mathbf{i}_D \sum_i a_i db_i = \sum_i a_i D(b_i)$$

One can prove that, if $\alpha \in \Omega_A^k$ and $D_0, ..., D_k \in \operatorname{Der} A$, then

$$\mathbf{i}_{D_0} ... \mathbf{i}_{D_n} d\alpha = \sum_{i=0}^{k} (-1)^i D_i \left(\mathbf{i}_{D_n} ... \mathbf{i}_{\widehat{D_i}} ... \mathbf{i}_{D_0} \alpha \right) + \sum_{i<j}^{0\cdots k} (-1)^{i+j} \mathbf{i}_{D_n} ... \mathbf{i}_{\widehat{D_j}} ... \mathbf{i}_{D_i} ... \mathbf{i}_{[D_i, D_j]} \alpha$$

We can also define the Lie derivative with Cartan magic formula

$$\mathcal{L}_D \alpha = d \mathbf{i}_D \alpha + \mathbf{i}_D d\alpha$$

and all the usual properties hold.

Now, if A is a Poisson algebra we can consider another “differential” calculus, which is based upon Poisson brackets instead of the associative product. More precisely, consider the space of Hamiltonian derivations $\operatorname{Ham} A =$

$\{X_a\}_{a\in A}$ and the module $\mathcal{H}_A$ generated by $\operatorname{Ham} A$: this module comes with a natural derivation $X\colon A \longrightarrow \mathcal{H}_A$ which extends $X\colon A \longrightarrow \operatorname{Ham}(A)$; whence, by the universal property of Kähler differentials, it exists an A-linear map (whose image will be $\mathcal{H}_A$)

$$\mathbf{H}\colon \Omega_A \longrightarrow \mathcal{H}_A$$

such that $\mathbf{H}\circ d = X$. For example, in the case of a symplectic Poisson algebra, this map is a module isomorphism, and $\mathcal{H}_A = \operatorname{Der} A = \hom_A(\Omega_A, A)$.

Now, basing us upon this Poisson differential X, we can construct a Poisson differential calculus as follows: consider the exterior powers $\mathcal{H}_A^k = \bigwedge^k \mathcal{H}_A$ and the graded extension of the differential X:

$$X(P \wedge Q) = XP \wedge Q + (-1)^{\deg P} P \wedge XQ$$

Notice that "forms" in Poisson settings are exterior products of derivations, which we call, according to a widely used terminology, *poly-derivations*: for example we can contract a Hamiltonian derivation on a derivation as

$$\mathbf{i}_D \sum_i a_i X_{b_i} = \sum_i a_i D(b_i)$$

This contraction is degenerate, and we are forced to consider the space

$$D_X = \{D \in \operatorname{Der} A \,|\, \forall c \in \operatorname{Cas} A \quad D(c) = 0\}$$

which is the space of derivations which act as zero on Casimir elements: in this way the contraction $\mathbf{i}\colon \mathcal{H}_A \times D_X \longrightarrow A$ is non degenerate.

Clearly this contraction extends to higher exterior powers, and also the morphism $\mathbf{H}$ extends, in a graded way, as

$$\mathbf{H}(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n) = (-1)^n \mathbf{H}(\omega_1) \wedge \mathbf{H}(\omega_2) \cdots \wedge \mathbf{H}(\omega_n)$$

where $\omega_i \in \Omega_A$. Notice that if we change the base ring of the algebra A from $\mathbb{K}$ to $\operatorname{Cas} A$ then we can transform this Poisson calculus in the usual differential calculus: indeed, be $A' = A \otimes_{\mathbb{K}} \operatorname{Cas} A$ (with the tensor product Poisson structure); then $\operatorname{Cas} A' = \operatorname{Cas} A$ so A' is a symplectic Poisson algebra over $\operatorname{Cas} A$ and $\mathcal{H}_{A'} \cong \Omega_{A'}$.

One can develop a Hamiltonian formalism upon the operator $\mathbf{H}$ (as Gel'fand and Dorfman did, cfr. [4]); first of all, notice that we have, by the universal property of Kähler differentials (which says in particular that $\operatorname{Der} A = \hom_A(\Omega_A, A)$), and Leibniz identity:

$$\{a, b\} = \langle \mathbf{H}(da), b\rangle = \mathbf{i}_{\mathbf{H}(da)} db$$

where $\langle\,\rangle$ is the pairing between derivations and differentials; this equation may be used to define Poisson brackets in terms of $\mathbf{H}$, as stated in the following theorem proved in [4]:

Theorem 4.1 *An associative algebra* $(A, \cdot)$ *is Poisson w.r.t. some brackets* $\{\,\}$ *if and only if there exists an operator* $\mathbf{H}\colon \Omega_A \longrightarrow \mathrm{Der}\, A$ *such that*

(1) $\{a, b\} = \langle \mathbf{H}(da), db\rangle$.

(2) $\langle \mathbf{H}\omega_1, \omega_2\rangle + \langle \omega_1, \mathbf{H}\omega_2\rangle = 0$.

(3) (3) $\langle \mathbf{H}\mathcal{L}_{\mathbf{H}\omega_1}\omega_2, \omega_3\rangle + \langle \mathbf{H}\mathcal{L}_{\mathbf{H}\omega_2}\omega_3, \omega_1\rangle + \langle \mathbf{H}\mathcal{L}_{\mathbf{H}\omega_3}\omega_1, \omega_2\rangle = 0$.

If we define a map $\pi\colon \Omega_A \wedge \Omega_A \longrightarrow A$ as

$$\pi(da, db) = \langle \mathbf{H}da, db\rangle$$

we get a tensor which is called *Poisson tensor* of the algebra A and that determines completely the brackets: to see it we must extend the usual Lie bracket in the space of derivation $\mathrm{Der}\, A$ to a bracket on the space of poly-derivations $\bigwedge^* \mathrm{Der}\, A$; this extension is guaranteed by the following result (cfr. [3], [9]):

Theorem 4.2 *There exists a unique bilinear operation*

$$[\,]\colon \bigwedge{}^i \mathrm{Der}\, A \times \bigwedge{}^j \mathrm{Der}\, A \longrightarrow \bigwedge{}^{i+j-1} \mathrm{Der}\, A$$

on the space $\bigwedge^* \mathrm{Der}\, A$ *such that*

(1) $[a, b] = 0, \qquad [a, D] = D(a), \qquad [D, D'] = DD' - D'D$

(2) $[P, Q] = (-1)^{pq}[Q, P]$

(3) $(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{r(q-1)}[R, [P, Q]] + (-1)^{q(p-1)}[Q, [R, P]] = 0$

(4) $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{q(p+1)} Q \wedge [P, R]$

for $a, b \in A$, $D, D' \in \mathrm{Der}\, A$ and $P \in \bigwedge^p \mathrm{Der}\, A$, $Q \in \bigwedge^q \mathrm{Der}\, A$, $R \in \bigwedge^r \mathrm{Der}\, A$.

Such brackets are called *Schouten–Nijenhuis brackets* and satisfy also the following useful formula

$$\mathbf{i}_{[P,Q]}\omega = (-1)^{q(p+1)} \mathbf{i}_P d\,(\mathbf{i}_Q \omega) + (-1)^p i_Q d\,(\mathbf{i}_P \omega) - \mathbf{i}_P \mathbf{i}_Q d\omega$$

where $P \in \bigwedge^p \mathrm{Der}\, A$, $Q \in \bigwedge^q \mathrm{Der}\, A$ and $\omega \in \Omega_A^{p+q-1}$.

Now we use these brackets (which are called *Schouten–Nijenhuis brackets*) to characterize Poisson brackets in terms of the tensor π (as noted by Lichnerowicz[4]):

[4] A coordinate version of this result was already known to Lie himself.

Theorem 4.3 *An associative algebra $(A, \cdot)$ is Poisson w.r.t. some brackets $\{\,\}$ if and only if it exists a tensor $\pi\colon \Omega_A \wedge \Omega_A \longrightarrow A$ such that*

(1) $\{a, b\} = \pi(da, db)$;

(2) $[\pi, \pi] = 0$.

As derivations play the rôle of forms in Poisson calculus, it is natural to ask for Lie brackets among them, as it happens for usual vector fields in Cartan calculus: and in fact one can prove that

Theorem 4.4 *There exists unique $\mathbb{K}$-Lie algebra brackets $\{\,\}$ on Ω_A such that:*

(1) $d\{a, b\} = \{da, db\}$;

(2) $\{\omega_1, a\omega_2\} = a\{\omega_1, \omega_2\} + \langle \mathbf{H}(\omega_1), da\rangle\omega_2$ *where* $\mathbf{H}\colon \Omega_A \longrightarrow \operatorname{Der} A$ *is the Hamiltonian operator induced by the Poisson structure on* A.

Again see [3] or [9] for a proof; from this it follows that

$$\mathbf{H}\{\omega_1, \omega_2\} = [\mathbf{H}\omega_1, \mathbf{H}\omega_2]$$

Moreover, by putting

$$\widetilde{X}(P) = -[\pi, P]$$

we can extend the operator $X\colon \bigwedge^k \mathcal{H}_A \longrightarrow \bigwedge^{k+1} \mathcal{H}_A$ to an operator

$$\widetilde{X}\colon \bigwedge{}^k \operatorname{Der} A \longrightarrow \bigwedge{}^{k+1} \operatorname{Der} A$$

where π is the Poisson tensor; this on $\mathcal{H}_A$ coïncides with X, and one may check that, for $P \in \bigwedge^p \operatorname{Der} A$ and $\omega_i \in \Omega_A$:

$$\langle \widetilde{X}(P), \omega_0 \wedge \ldots \wedge \omega_p\rangle = \sum_{i=0}^{p} (-1)^i \pi(\omega_i, d\mathbf{i}_P \omega_1 \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \wedge \omega_p) + \\ + \sum_{i<j}^{0\ldots p} (-1)^{i+j} \mathbf{i}_P\{\omega_i, \omega_j\} \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \widehat{\omega_j} \wedge \ldots \wedge \omega_p$$

Of course, by the usual computations, this formula implies that

$$\widetilde{X} \circ \widetilde{X} = 0$$

So we have a cochain complex $(\bigwedge^* \operatorname{Der} A, \widetilde{X})$ and a natural subcomplex $(\bigwedge^* \mathcal{H}_A, X)$: they give rise to two cohomologies; the former is usually called

Poisson cohomology and was defined by Lichnerowicz in 1977 (in the differential geometrical setting), the latter has no name (I call it *symplectic cohomology* for geometrical reasons). Notice that the condition on π to be a Poisson tensor becomes a cocycle condition: $\widetilde{X}\pi = 0$; in any case, a Poisson tensor defines a cohomology class both in the H^2 of the Poisson and in the symplectic cohomology.

These cohomology spaces are actually algebras: indeed X is a graded derivation, so that we can define a *cap* product as $[P] \cap [Q] = [P \wedge Q]$. But then, since one can prove that

$$\widetilde{X} \circ \mathbf{H} + \mathbf{H} \circ d = 0$$

(being d the usual differential of the de Rham complex for Kaḧler differentials), we get a map

$$\mathbf{H}^*\colon H^*_{dr}(A) \longrightarrow H^*_\pi(A)$$

from de Rham to Poisson cohomology: for example, if the algebra A is symplectic, this morphism is an isomorphism of DG-algebras.

Example 4.5 *$H^0_\pi(A) = Cas\,A$: indeed $\bigwedge^0 Der\,A = A$ and being $a \in \bigwedge^0 Der\,A$ a cocycle means that $\widetilde{X}(a) = 0$, thus $X_a = 0$ so that, for each $b \in A$: $\{a, b\} = 0$ and hence a is Casimir. The same is true for symplectic cohomology.*

Example 4.6 *$H^1_\pi(A) = Can\,A/Ham\,A$ (remember that $Can\,A$ are $\mathbb{K}$-linear operators $A \longrightarrow A$ which are both derivations for the associative and for the Lie algebra A); in fact a 1-cocycle is an element $D \in Der\,A$ such that $\widetilde{X}(D) = 0$, i.e., for each $a, b \in A$:*

$$\begin{aligned} 0 &= \langle \widetilde{X}D, da \wedge db\rangle = \langle \mathbf{H}(da), dD(b)\rangle - \langle \mathbf{H}(db), dD(a)\rangle - \langle D, \{da, db\}\rangle \\ &= \{D(a), b\} + \{a, D(b)\} - D\{a, b\} \end{aligned}$$

whence 1-cocycles are precisely canonical derivations; on the other hand it is obvious that Hamiltonian derivations are 1-coboundaries, so that $H^1_\pi(A) = Can\,A/Ham\,A$.

One can also interpret $H^2_\pi(A)$ in terms of deformations of Poisson structures.

Example 4.7 *If A is a Lie–Poisson algebra, thus $A = Sym\,(\mathfrak{g}^*)$, then Poisson cohomology is the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the representation A (this was proved geometrically by Ginzburg, Lu and Weinstein).*

We only make a mention of the fact that also a Poisson homology does exist, as defined by Koszul (cfr. [6]) but it is not really the dual of Poisson cohomology: the search for such a dual it is yet not concluded.

5 Modules on Poisson algebras

When facing the problem of defining a notion of module over a Poisson algebra different choices arise: I propose the following

Definition 5.1 *A* Poisson module E *over a Poisson algebra* $(A,\cdot,\{\,\})$ *is both a module over the algebra* $(A,\cdot)$ *and a representation of the Lie algebra* $(A,\{\,\})$ *such that*

$$\{a,b\}\cdot e = a\{b,e\} - \{b, a\cdot e\}$$

where $a,b\in A$, $e\in E$, $\cdot$ *means the action of* $(A,\cdot)$ *on* E *and* $\{\,\}$ *means the action of* $(A,\{\,\})$ *on* E.

For example both A and A' (the dual vector space of A) are Poisson modules w.r.t. adjoint and coadjoint actions.

Example 5.2 *The module of derivations* $\operatorname{Der} A$ *is Poisson by means of*

$$(a\cdot D)(b) = a\cdot D(b) \qquad \text{and} \qquad \{a,D\} = [X_a, D]$$

Indeed the associative action is the adjoint one, while

$$\begin{aligned}\{\{a,b\},D\} =& [X_{\{a,b\}},D] = [[X_a,X_b],D] = [X_a,[X_b,D]] - [X_b,[X_a,D]]\\ =& \{a,\{b,D\}\} - \{b,\{a,D\}\}\end{aligned}$$

and

$$\{a,bD\} = [X_a,bD] = b[X_a,D] + \{a,b\}D = b\{a,D\} + \{a,b\}D$$

so the Lie action is Poisson.

Also the A-module $\mathcal{H}_A$ generated by Hamiltonian derivations is Poisson w.r.t. these actions: we only need to check that if $D\in\mathcal{H}_A$ then $\{a,D\}\in\mathcal{H}_A$ and, since $D=\sum_i a_i X_{b_i}$ we have

$$\{a,a_iX_{b_i}\} = [X_a,a_iX_{b_i}] = a_i[X_a,X_{b_i}] + \{a,a_i\}X_{b_i} = a_iX_{\{a,b_i\}} + \{a,a_i\}X_{b_i}$$

which again belongs to $\mathcal{H}_A$.

Definition 5.3 *A* multiplicative module E *over a Poisson algebra* $(A,\cdot,\{\,\})$ *is both a module over the algebra* $(A,\cdot)$ *and a representation of the Lie algebra* $(A,\{\,\})$ *such that*

$$\{ab,e\} = a\{b,e\} + b\{a,e\}$$

where $a,b\in A$, $e\in E$, $\cdot$ *means the action of* $(A,\cdot)$ *on* E *and* $\{\,\}$ *means the action of* $(A,\{\,\})$ *on* E.

The concept of a Poisson module is different from that of a multiplicative module: for example Der A is not a multiplicative module, since

$$\begin{aligned}\{ab, D\} &= [X_{ab}, D] = [aX_b, D] + [bX_a, D] \\ &= a\{b, D\} + b\{a, D\} - D(a)X_b - D(b)X_a\end{aligned}$$

Example 5.4 *The module Ω_A of Kähler differentials on a Poisson algebra A is a Poisson module but not a multiplicative one w.r.t. the Poisson action*

$$\{a, \omega\} = \mathcal{L}_{X_a}\omega = d\mathbf{i}_{X_a}\omega + \mathbf{i}_{X_a}d\omega$$

Indeed this defines a Lie action

$$\begin{aligned}\{\{a, b\}, \omega\} &= \mathcal{L}_{[X_a, X_b]}\omega = [\mathcal{L}_{X_a}, \mathcal{L}_{X_b}]\omega \\ &= \mathcal{L}_{X_a}\{b, \omega\} - \mathcal{L}_{X_b}\{a, \omega\} = \{a, \{b, \omega\}\} - \{b, \{a, \omega\}\}\end{aligned}$$

which is Poisson

$$a\{b, \omega\} - \{b, a\omega\} = a\mathcal{L}_{X_b}\omega - \{b, a\}\omega - a\mathcal{L}_{X_b}\omega = \{a, b\}\omega$$

Notice that

$$\{a, \omega\} = \{da, \omega\}$$

where brackets on rhs of this equations represents the commutator between differential forms given by Theorem 4.4.

This example suggests how to construct a general class of Poisson modules: consider a representation E of the Lie algebra ω_A which is also an A-module and such that

$$[\omega, ae] = a[\omega, e] - \mathbf{i}_{X_a}\omega e$$

where $[\,]$ denotes the action of Ω_A on E. If we define

$$\{a, e\} = [da, e]$$

it's easy to see that we get a Poisson structure on E.

For such Poisson modules we can consider Chevalley–Eilenberg cohomology of the Lie algebra Ω_A with coefficients in E and we find that

Theorem 5.5 *Poisson cohomology is precisely the cohomology of the Lie algebra Ω_A with coefficient in the representation A with its Poisson module structure.*

Example 5.6 *If $E = \mathrm{Der}\,A$ and if we put*

$$[\omega, D] = [\mathbf{H}\omega, D]$$

(where $\mathbf{H}$ *is the operator given in Theorem 4.1) we find that the Poisson module structure on* $\mathrm{Der}\,A$ *we discussed above can be obtained as* $\{a, D\} = [da, D]$.

But notice, for example, that the Poisson structure on A' can't be induced by any representation of Ω_A.

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